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The convex structure of the set of positive approximants for a given operator

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§ 1. Introduction

In [6], P. R. HALMOS showed that any (bounded linear) operator T has a positive approximant, denoted by P_0. This means that P_0 is a nonnegative operator and the norm $\|T-P_0\|$ is the same as the distance from T to the set of nonnegative operators. Other basic facts were collected in [6] and in [2]. Halmos asked for the extreme points of the convex set of positive approximants, denoted by $\mathscr{P}(T)$, and for a characterization of those T for which $\mathscr{P}(T)$ is a singleton set. This paper characterizes a normal operator T for which $\mathscr{P}(T)$ is q-dimensional and constructs some extreme points of that set. (For dimension of a convex set see pp. 7—9 of [8].)

In [3] we studied the set of positive near-approximants of T, denoted $\mathscr{P}'(T)$, where a positive near-approximant is a best approximation for T using the new norm

$$|||T|||^2 = \|B^2+C^2\|$$

with $T=B+iC$, $B=B^*$, $C=C^*$. The distance from T to the nonnegative operators is the same whether it is computed with the new norm or with the operator norm. This distance is denoted by $\delta(T)$ and is referred to as the *modulus of positivity*. Recall from [3] that the new norm is between the operator norm and the numerical radius. We use [5] as a source for many terms and facts that we shall not explain.

§ 2. Preliminaries

Frequently in the study of a convex set the dimension of the convex set is apparent. Then generally the investigation turns to the subtler question of determinig the extreme points of the convex set. Moreover, if the nonempty convex set is a compact subset of some locally convex topological vector space then the Krein—Milman theorem implies that the closed convex hull of the extreme points of the convex set is the convex set. In the case that the dimension of either $\mathscr{P}(T)$ or $\mathscr{P}'(T)$ is finite then the following theorem should inspire some interest in the extreme points.

2.1. Theorem. *Both of the convex sets $\mathscr{P}(T)$ and $\mathscr{P}'(T)$ are the closed convex hull of their extreme points.*

Proof. Clearly the set of bounded operators on H is a locally convex topological vector space with the weak operator topology and so it suffices to show that each of the two sets is compact in the weak operator topology. Since any ball of bounded operators is compact in the weak operator topology (see problem 6 p. 512 [4]) and since both sets $\mathscr{P}(T)$ and $\mathscr{P}'(T)$ are obviously contained in such balls, it would suffice to show that both sets are closed in the weak operator topology. We define two functions φ and ψ on the bounded operators of H by the formulas

$$\varphi(S) = \|T-S\|, \quad \psi(S) = |||T-S|||.$$

Let $\mathscr{R}=\{S:\varphi(S)>\delta(T)\}$ and $\mathscr{S}=\{S:\psi(S)>\delta(T)\}$. Since the nonnegative operators are obviously closed in the weak operator topology it would suffice to prove that $\mathscr{R}$ and $\mathscr{S}$ are open in that topology. In order to prove the last assertion it would suffice to show that φ and ψ are lower semicontinuous and it is clear that φ can be written as the supremum of functions which are obviously continuous with respect to the weak operator topology. It follows that $\mathscr{P}(T)$ is compact in the weak operator topology and the same conclusion will follow for $\mathscr{P}'(T)$ once we show that ψ is lower semicontinuous. The last property can be deduced from the following formula from Theorem 3.1 of [3]:

$$|||T||| = \frac{1}{2}\|T^*T+TT^*\|^{1/2} = \frac{1}{2}[w(T^*T+TT^*)]^{1/2}.$$

Any consideration of the convex structure of either $\mathscr{P}(T)$ or $\mathscr{P}'(T)$ will require the following theorem from [6].

2.2. Theorem. (Halmos) *If $T=B+iC$ with $B=B^*$, $C=C^*$ then*

$$\inf\{\|T-P\|:P\geqq 0\} = \inf\{r:B+(r^2-C^2)^{1/2}\geqq 0\}.$$

If the above quantity is denoted by δ then $P_0=B+(\delta^2-C^2)^{1/2}$ is a positive approximant for T.

§ 3. The main theorem

The main theorem of this paper will be proved with a sequence of lemmas in the next section. In this section we state the result.

3.1. Theorem. *Let $\mathscr{P}'(T)$ denote the convex set of positive near-approximants for the normal operator T and let*

$$H_0 = (P_0H)^- \cap ((\delta^2-C^2)H)^-.$$

If p is the dimension of H_0 then

$$\dim \mathscr{P}'(T) = p^2.$$

Here all infinite cardinal numbers are identified.

After this theorem is proved the techniques will be extended to obtain the same result for $\mathscr{P}(T)$. Then by restricting the generality we shall construct extreme points for both sets. However, we should note that a positive near-approximant is not necessarily a positive approximant. For example, the positive part of the real part of $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is a positive near-approximant although the Halmos positive approximant P_0 is the unique positive approximant.

In the sequel we shall repeatedly need the following result from [3].

3.2. Lemma. *If $R_1, \dots, R_n$ are commuting nonnegative operators on H then there is a nonnegative operator R such that*

(i) $RR_j = R_jR$ *for every* j,

(ii) $R \leqq R_j$ *for every* j,

(iii) $(RH)^- = \cap \{(R_jH)^- : j = 1, \dots, n\}$.

§ 4. Proof of the main theorem

Notations. By A_0 we shall denote the positive operator constructed from P_0 and $2(\delta^2 - C^2)^{1/2}$ by appeal to Lemma 3.2 with T normal. Thus A_0 is dominated by P_0 and $2(\delta^2 - C^2)^{1/2}$; A_0 commutes with both operators and $(A_0H)^-$ is H_0.

4.1. Lemma. *If A is a positive operator such that*

$$0 \leqq A \leqq A_0$$

and A commutes with $(\delta^2 - C^2)^{1/2}$ then $P_0 - A$ is a positive near-approximant for T.

Proof. In view of the given inequality we have

$$0 \leqq A \leqq P_0 \quad \text{and} \quad A \leqq 2(\delta^2 - C^2)^{1/2}.$$

Thus we have the following inequality

$$-(\delta^2 - C^2)^{1/2} \leqq A - (\delta^2 - C^2)^{1/2} \leqq (\delta^2 - C^2)^{1/2}$$

and consequently we have

$$(\delta^2 - C^2)^{1/2} \pm \left(A - (\delta^2 - C^2)^{1/2}\right) \geqq 0.$$

Since A commutes with $(\delta^2 - C^2)^{1/2}$ the above two inequalities imply that

$$(\delta^2 - C^2) - \left(A - (\delta^2 - C^2)^{1/2}\right)^2 \geqq 0,$$

i.e.

$$\delta^2 \geqq \left(A - (\delta^2 - C^2)^{1/2}\right)^2 + C^2.$$

It now follows that

$$\delta^2 \geqq \|(A-(\delta^2-C^2)^{1/2})^2+C^2\| = \||A-(\delta^2-C^2)^{1/2}+iC|\|^2 =$$
$$= \||A+T-P_0|\|^2 = \||T-(P_0-A)|\|^2$$

and so P_0-A is a positive near-approximant for T.

4.2. Lemma. *Let $E(\cdot)$ be the spectral measure for a self adjoint operator A. If $E([a, b))\neq 0$ then either* (i) *or* (ii) *below holds:*

(i) $E([a, c))\neq 0$ and $E([c, b))\neq 0$ for *some* $c\in(a, b)$
(ii) $\sigma(A|E([a, b))H)=\{e\}$ for some $e\in[a, b)$.

Proof. Take two strictly monotone sequences, say $\{a_k\colon k=0, 1, \ldots\}$ and $\{b_j\colon j=0, 1, \ldots\}$, such that $a_0=(a+b)/2=b_0$, $a_k\to a$, $b_j\to b$ and $|a_k-a_{k+1}|<\frac{1}{2}$, $|b_j-b_{j+1}|<\frac{1}{2}$ for $j, k=0, 1, \ldots$. Since

$$(*)\qquad [a, b) = \{a\}\cup\bigcup_{k=0}^{\infty}[a_{k+1}, a_k)\cup\bigcup_{j=0}^{\infty}[b_j, b_{j+1}),$$

at least one of the sets on the right of $(*)$ has nonzero measure. If only $\{a\}$ has nonzero measure then $E((a, b))=0$ and so

$$\sigma(A|E([a, b))H) = \sigma(A|E(\{a\})H)\subset\{a\}$$

which proves (ii) above since no bounded operator can have empty spectrum. If two sets on the right of $(*)$ have nonzero measure then clearly (i) follows from an appropriate choice of c. Thus we may assume that exactly one interval on the right of $(*)$ has nonzero measure; denote that interval by $[c_1, d_1)$. Partition this interval in a manner analogous to $(*)$ except that every subinterval has length less than $\frac{1}{3}$. As reasoned above, either the lemma is proved at this step or else there is exactly one subinterval, say $[c_2, d_2)$, with nonzero measure. Either this process terminates and the lemma is proved or else it continues indefinitely. Assume the latter and let the intervals constructed be $\{[a\ b), [c_1, d_1), [c_2, d_2), \ldots\}$. Thus the sequence $\{E([a, b))$ $E([c_1, d_1)), E([c_2, d_2)), \ldots\}$ consists of only one nonzero constant. By the Monotone Convergence Theorem (applied weakly) that constant is the measure of

$$S = \bigcap_{j=1}^{\infty}[c_j, d_j).$$

By the construction of the subintervals S consists of only one point, say $S=\{e\}$. Thus

$$\sigma(A|E([a, b))H) = \sigma(A|E(\{e\})H)\subset\{e\}$$

and equality follows since the bounded operator cannot have empty spectrum.

4.3. Lemma. *Let $\{Q_1, Q_2, \ldots\}$ be a countably infinite set of mutually orthogonal nonzero projections onto subspaces of A_0H which reduce A_0. Then the set $\mathscr{C}=\{A_0Q_k: k=1, 2, \ldots\}$ is linearly independent over the real numbers.*

Proof. Assume that $c_1, \ldots, c_m$ are constants such that $\sum_{j=1}^{m} c_j A_0 Q_{n(j)}=0$ or equivalently

$$0 = A_0 \sum_{j=1}^{\infty} c_j Q_{n(j)}.$$

Since $Q_{n(j)}H\subset A_0H$ and A_0 is one-to-one on A_0H, we see that $A_0Q_{n(j)}\neq 0$. Furthermore, since the projections are mutually orthogonal, for each j we can choose a vector f such that $Q_{n(j)}f\neq 0$. Thus every c_j is zero and this proves the linear independence of $\mathscr{C}$.

4.4. Lemma. *If A_0H is infinite dimensional and the spectrum of A_0 is not a finite set then the convex set of all positive near-approximants of T is infinite dimensional.*

Proof. Let $\mathscr{B}$ consist of all collections of disjoint intervals, for example $\mathscr{S}=\{I_\gamma: \gamma\in\Gamma\}$, where $I_\gamma=[a_\gamma, b_\gamma)$ and $E(I_\gamma)\neq 0$ for all $\gamma\in\Gamma$ and $E(\cdot)$ is the spectral measure of A_0. Then $\mathscr{B}$ is partially ordered by inclusion and Zorn's lemma easily shows the existence of a maximal element $\mathscr{S}_0$. For the sake of obtaining a contradiction assume that $\mathscr{S}_0$ is finite, say $\mathscr{S}_0=\{I_1, \ldots, I_m\}$ with $I_k=[a_k, b_k)$ for $k=1, \ldots, m$. If we could find $c_k\in(a_k, b_k)$ for some k such that

$$E([a_k, c_k)) \neq 0 \quad \text{and} \quad E([c_k, b_k)) \neq 0$$

then we would contradict the maximality of $\mathscr{S}_0$. Thus we may appeal to Lemma 4.2 and conclude that there exists $e_k\in I_k$ for $k=1, \ldots, m$ such that

$$\sigma(A_0|E(J)H)\subset\{e_1, \ldots, e_m\}$$

where $J=\bigcup_{k=0}^{m} I_k$. Since

$$\sigma(A_0) = \sigma(A_0|E(J)H)\cup\sigma(A_0|E(J')H)$$

with $J'=[0, 2\,\|A_0\|)\setminus J$ and since the spectrum of A_0 is not a finite set, it follows that $\sigma(A_0|E(J')H)$ is nonempty and $E(J')$ is nonzero. From the form of the intervals I_k and the definitions of J and of J' it is clear that we can find an interval $[\alpha, \beta)$ contained in J' with the property that $E([\alpha, \beta))\neq 0$. This contradicts the maximality of $\mathscr{S}_0$ and consequently $\mathscr{S}_0$ must be infinite. Let $\mathscr{S}_1$ be a countably infinite collection of intervals belonging to $\mathscr{S}_0$ with the property that zero does not belong to any interval, say $\mathscr{S}_1=\{I_1, I_2, \ldots\}$.

Let $\mathscr{C}=\{A_0E(I_n): n=1, 2, \ldots\}$. Because $(\delta^2-C^2)^{1/2}$ commutes with A_0 it commutes with $E(I_n)$ for $n=1, 2, \ldots$ and consequently Lemma 4.1 shows that each element of $\{P_0 - C: C \in \mathscr{C}\}$ is a positive near-approximant. Since zero does not belong to I_n we have $E(I_n)H \subset A_0 H$ and Lemma 4.3 shows that $\mathscr{C}$ is linearly independent over the reals. If $\mathscr{H} + S$ is any translate of a vector space $\mathscr{H}$ such that $\mathscr{H} + S \supset \mathscr{P}'(T)$ then 0 belongs to $\mathscr{H} + S - P_0$. Thus $S - P_0$ belongs to $\mathscr{H}$ thus $\mathscr{H} + S - P_0$ is actually $\mathscr{H}$ and so $\mathscr{H} \supset -\mathscr{C}$ consequently

$$\dim \mathscr{H} \geqq \dim \mathscr{C}.$$

Hence $\mathscr{P}'(T)$ is an infinite dimensional convex set.

4.5. Lemma. *Let $\mathscr{P}'(T)$ denote the set of positive near-approximants of $T = B + iC$ with $B = B^*$, $C = C^*$. If the dimension of H_0 is a finite positive integer p then the dimension of the convex set $\mathscr{P}'(T)$ is not greater than p^2.*

Proof. An operator on H_0 is self adjoint if its matrix (with respect to any basis) is conjugate symmetric; this fact can be used to give a basis for the self adjoint operators on H_0 considered as a real vector space. The number of elements in that basis is p^2, and consequently by the argument in the last two sentences of the proof of Lemma 4.4 it would suffice to prove that

$$(P_0 - P)H \subset H_0$$

for every $P \in \mathscr{P}'(T)$. According to Corollary 3.2 of [3] we know that $P_0 \geqq P_0 - P \geqq 0$ and so $\ker P_0 \subset \ker (P_0 - P)$, consequently $((P_0 - P)H)^- \subset (P_0 H)^-$ for every $P \in \mathscr{P}'(T)$ Clearly it would suffice to prove that $((P_0 - P)H)^-$ is contained in $((\delta^2 - C^2)^{1/2} H)^-$. Because

$$\delta^2 = |||T - P|||^2 = \|(B - P)^2 + C^2\|$$

thus $\delta^2 - C^2 \geqq (B - P)^2$, and so we have

$$(\delta^2 - C^2)^{1/2} \geqq |B - P| \geqq B - P,$$

consequently

$$2(\delta^2 - C^2)^{1/2} \geqq P_0 - P.$$

By the above argument the lemma is proved.

4.6. Lemma. *The convex set $\mathscr{P}'(T)$ is infinite dimensional over the reals if and only if H_0 is an infinite dimensional subspace.*

Proof. Recall that the operator A_0 constructed by appeal to Lemma 3.2 has the property that

$$(*) \qquad (A_0 H)^- = H_0$$

and so the hypothesis implies that $A_0 H$ is infinite dimensional. The case that the spectrum of A_0 is not a finite set is handled by Lemma 4.4 and consequently we may assume that

$$\sigma(A_0) = \{\lambda_1, \ldots, \lambda_p\}.$$

Since A_0 is self adjoint it is easily seen that each λ_j is an eigenvalue and

$$A_0 H = (A_0 H)^- = E(\{\lambda_1', \ldots, \lambda_e'\})H$$

where $\{\lambda_1', \ldots, \lambda_e'\}$ is the set of nonzero eigenvalues of A_0 and $E(\cdot)$ is the spectral measure of A_0. Thus one of the nonzero eigenvalues, say λ_i', has infinite multiplicity.

Because P_0 and $(\delta^2-C^2)^{1/2}$ commute with A_0 they commute with $E(\{\lambda_1'\})$; $E(\{\lambda_1'\})H$, which we shall denote by H_1, reduces P_0 and $(\delta^2-C^2)^{1/2}$. In view of $(*)$ we may assume that $(\delta^2-C^2)^{1/2}H_1$ is infinite dimensional. Thus if the spectrum of $(\delta^2-C^2)^{1/2}|H_1$ is finite then it has a nonzero eigenvalue with infinite multiplicity. Clearly we can find a countably infinite collection of mutually orthogonal projections onto subspaces of H_1 which reduce $(\delta^2-C^2)^{1/2}|H_1$, say $Q_1, Q_2, \ldots$. Since $Q_k H$ reduces A_0 and is contained in $A_0 H_1$ for $k=1, 2, \ldots$, we see by Lemma 4.3 that $\mathscr{C}=\{A_0 Q_k\colon k=1, 2, \ldots\}$ is a linearly independent set over the reals. Because Q_k commutes with $(\delta^2-C^2)^{1/2}$ we conclude from Lemma 4.1 each element of $\{P_0-C\colon C\in\mathscr{C}\}$ is a positive near-approximant. Thus in the case that $(\delta^2-C^2)^{1/2}|H_1$ has finite spectrum $\mathscr{C}$ is an infinite set of linearly independent positive operators.

If $(\delta^2-C^2)^{1/2}|H_1$ does not have finite spectrum then we can exploit Lemma 4.2 as was done in the first paragraph of the proof of Lemma 4.4 to obtain a countably infinite set of mutually orthogonal projections onto subspaces of H_1 which reduce $(\delta^2-C^2)^{1/2}$. As in the paragraph above we deduce the existence of an infinite set of positive operators which is linearly independent over the reals. By the argument in the last two sentences of the proof of Lemma 4.4, we have shown $\mathscr{P}'(T)$ to be an infinite dimensional convex set provided that H_0 is an infinite dimensional subspace.

If H_0 were a nontrivial finite dimensional subspace then we could conclude from Lemma 4.5 that $\mathscr{P}'(T)$ is a finite dimensional convex set. If H_0 were trivial then Theorem 4.2 in [3] implies that $\mathscr{P}'(T)=\{P_0\}$. Hence this lemma is proved.

4.7. Lemma. *The subspace H_0 is finite dimensional then it reduces the operator $\left(\frac{1}{2}A_0-(\delta^2-C^2)^{1/2}\right)^2+C^2$; if in addition S denotes the restriction of that operator to H_0 then $\|S\|<\delta^2$.*

Proof. Since $(A_0H)^-=H_0$ and H_0 is finite dimensional, we have $A_0H=H_0$. Because A_0 commutes with C^2 we see that H_0 is invariant under C^2 and $(\delta^2-C^2)^{1/2}$; thus H_0 reduces $\left(\frac{1}{2}A_0-(\delta^2-C^2)^{1/2}\right)^2+C^2$.

Note that Lemma 4.1 implies that $P_0-\frac{1}{2}A_0$ is a positive near-approximant for T and consequently we know that

$$\|S\| = \left\|\left(T-P_0+\frac{1}{2}A_0\right)\Big|H_0\right\|^2 \leqq \delta^2.$$

If equality holds in the above inequality then δ^2 is an eigenvalue of S. In order to obtain a contradiction we assume this. Let $H_1=\ker(S-\delta^2)$ and take $0\neq h\in H_1$. By the commutativity of A_0 and C^2 we have

$$\left(\frac{1}{4}A_0^2+A_0(\delta^2-C^2)^{1/2}+(\delta^2-C^2)\right)h=\left(\frac{1}{2}A_0-(\delta^2-C^2)^{1/2}\right)^2 h=(\delta^2-C^2)h$$

or $\left(\frac{1}{4}A_0-(\delta^2-C^2)^{1/2}\right)A_0h=0$. Since $\frac{1}{4}A_0-(\delta^2-C^2)^{1/2}$ is obviously invertible, it must be that $h\in\ker A_0$. On H_0 the operator A_0 is one-to-one and so $h=0$. This contradiction proves the lemma.

4.8. Lemma. *If the dimension of H_0 is the finite positive integer p then*

$$\dim \mathscr{P}'(T)=p^2.$$

Proof. It was established in Lemma 4.5 that p^2 is an upper bound for the dimension of $\mathscr{P}'(T)$. Recall that A_0 commutes with both P_0 and $(\delta^2-C^2)^{1/2}$; also we have $(A_0H)^-=H_0$. By the finite dimensionality we have $A_0H=H_0$ and consequently H_0 is invariant under P_0 and $(\delta^2-C^2)^{1/2}$; thus H_0 reduces A_0 and $(\delta^2-C^2)^{1/2}$ and we can simultaneously diagonalize $A_0|H_0$ and $(\delta^2-C^2)^{1/2}|H_0$. Let $\{e_1, \dots, e_p\}$ be an orthonormal basis which simultaneously diagonalizes the two restrictions above and let Q_k be the orthogonal projection onto e_k for $k=1, \dots, p$. Note that each $Q_k|H_0$ commutes with both $A_0|H_0$ and $(\delta^2-C^2)^{1/2}|H_0$ and $Q_kH\subset A_0H$. By Lemma 4.1 $\{P_0-A_0Q_k\colon k=1, \dots, p\}$ consists of positive near-approximants and the argument used to prove Lemma 4.3 shows that $\{A_0Q_k\colon k=1, \dots, p\}$ is a linearly independent set.

Since H_0 reduces $(\delta^2-C^2)^{1/2}$ it clearly reduces C^2 and we may apply Lemma 4.7. Assume that A_0 restricted to H_0 is diag $\{2a_1, \dots, 2a_p\}$ with $a_1\geqq a_2\geqq\dots\geqq a_p$ and note that a_p is positive since $H_0=A_0H$. For any positive γ not greater than a_p define A_γ on H_0 by

$$A_\gamma=\operatorname{diag}\{a_1-\gamma, \dots, a_p-\gamma\}+\langle\cdot, e_k\rangle\gamma e_j+\langle\cdot, e_j\rangle\gamma e_k$$

for any pair of $j, k=1, \dots, p$ and $k>j$. Since A_γ is obviously self adjoint and converges to $\frac{1}{2}A_0$ in operator norm, the upper semicontinuity of the spectrum of $\frac{1}{2}A_0$ shows that A_γ is nonnegative for all γ sufficiently small. Because the norm of

$$\langle\cdot, e_k\rangle e_j+\langle\cdot, e_j\rangle e_k$$

is one, it is easy to see that $\frac{1}{2}A_0-A_\gamma$ is nonnegative. The continuity of the expression

$$(X-(\delta^2-C^2)^{1/2})^2+C^2$$

in X with respect to the operator norm and Lemma 4.7 show that

$$\|(A_\gamma-(\delta^2-C^2)^{1/2})^2+C^2\| \leqq \delta^2$$

for all γ sufficiently small. It now follows that for all γ sufficiently small we have

$$0 \leqq A_\gamma \leqq \frac{1}{2}A_0 \leqq P_0$$

$$|||T-(P_0-A_\gamma)|||^2 = \|(A_\gamma-(\delta^2-C^2)^{1/2})^2+C^2\| \leqq \delta^2$$

where A_γ has been extended to all of H by making it zero on the orthogonal complement of H_0. Thus P_0-A_γ is a positive near-approximant of T. An analogous argument shows that $P_0-A'_\gamma$ is a positive near-approximant when A'_γ is zero on $(H_0)^\perp$ and its restriction to H_0 is

$$\operatorname{diag}\{a_1-\gamma, \ldots, a_p-\gamma\} \;+\; \langle\cdot, e_k\rangle i\gamma e_j \;-\; \langle\cdot, e_j\rangle i\gamma e_k$$

and γ is sufficiently small. The linear independence over the reals of the following set is apparent:

$$\{A_0Q_i, A_\gamma, A'_\gamma\colon\ i=1, \ldots, p \text{ and } k>j\}.$$

There are p^2 operators in this set and the argument in the last two sentences of the proof of Lemma 4.4 shows that the dimension of $\mathscr{P}'(T)$ as a convex set is at least p^2. Equality then follows from Lemma 4.5.

Proof of Theorem 3.1. This theorem follows from Lemma 4.6 if p is infinite; it follows from Lemma 4.8 if p is a finite positive integer; it follows from Theorem 4.2 of [3] if p equals zero.

§ 5. Consequences of § 4 for positive approximants

The method used in the preceding section to construct positive near-approximants was initiated as a method for constructing positive approximants in Theorem 4.3 of [2]. The object of this section is to show that the construction of near-approximants in the preceding section can be refined so that approximants result. First we must prove a result analogous to Lemma 4.7 which can be applied to the operator norm just as Lemma 4.7 was applied to the new norm.

5.1. Lemma. *If A_0 commutes with C and H_0 is finite dimensional then H_0 reduces $\frac{1}{2}A_0-(\delta^2-C^2)^{1/2}+iC$ and if S denotes the restriction of that operator to H_0 then we have $\|S\|<\delta$.*

Proof. Since $(A_0H)^- = H_0$ and H_0 is finite dimensional, we have $A_0H = H_0$. Because A_0 commutes with C we know that H_0 reduces C, $(\delta^2-C^2)^{1/2}$ and $\frac{1}{2}A_0-$ $-(\delta^2-C^2)^{1/2}+iC$. Moreover, the last operator is normal and consequently S is normal. Because the numerical radius of S equals $\|S\|$, Theorem 3.1 of [3] implies that $|||S|||$ equals $\|S\|$. The desired conclusion now follows from Lemma 4.7.

Now we give our basic theorem for positive approximants.

5.2. Theorem. *Let T be a normal operator and let p be the dimension of the subspace H_0. If $\mathscr{P}(T)$ denotes the convex set of positive approximants of T then its real dimension is p^2. Here all infinite cardinal numbers are identified.*

Proof. By Lemma 4.5 it is immediate that the dimension of $\mathscr{P}(T)$ is not greater than p^2; recall that $\mathscr{P}(T)$ is contained in $\mathscr{P}'(T)$. The equality will be established when we show that each positive near-approximant previously constructed is in fact a positive approximant. Note that each positive near-approximant constructed in the proofs of Lemma 4.4, Lemma 4.6 and the first part of the proof of Lemma 4.8 has form P_0-A_0Q where Q is an orthogonal projection commuting with A_0 and $(\delta^2-C^2)^{1/2}$. Because T is normal, B and C commute. Furthermore, it is straightforward to see that Lemma 3.2 implies that there is a positive operator A_1 dominated by $\sqrt{2}(\delta-C)^{1/2}$, $\sqrt{2}(\delta+C)^{1/2}$ and $P_0^{1/2}$; clearly A_1 commutes with C and with P_0. If we set $A_0=(A_1)^2$ then it is routine to see that this A_0 has all the properties of the previous A_0 and also it commutes with C and B. It follows that $\{A_0, B, C, P_0\}$ is a set of commuting operators and consequently each commutes with all spectral projections of the others. If we take Q to be a spectral projection for one of the above operators then

$$\|T-(P_0-A_0Q)\|^2 = \|-(\delta^2-C^2)^{1/2}+iC+A_0Q\|^2 =$$
$$= |||-(\delta^2-C^2)^{1/2}+iC+A_0Q|||^2 \leqq \delta^2$$

since both norms agree on normal operators. (Recall Theorem 3.1 of [3].) An examination of those previous constructions shows that Q can be taken to be such a spectral projection of either A_0 or $(\delta^2-C^2)^{1/2}$ except possibly when $(\delta^2-C^2)^{1/2}$ restricted to an infinite dimensional eigenspace of A_0 has an infinite dimensional eigenvalue. In the latter case restrict P_0 to the infinite dimensional eigenspace of $(\delta^2-C^2)^{1/2}$ and use either the spectral projections or else projections onto arbitrary eigenvectors of P_0. Thus if H_0 is infinite dimensional it is established that $\mathscr{P}(T)$ is infinite dimensional. Of course, if H_0 is $\{0\}$ then $\mathscr{P}'(T)$ is just $\{P_0\}$ by Theorem 4.2 of [3] and necessarily $\mathscr{P}(T)=\{P_0\}$ which proves this theorem in that case.

The only remaining case requires that p be a finite positive integer. The first positive near-approximants constructed in Lemma 4.8 have the form P_0-A_0Q and we may use the argument above to guarantee that each P_0-A_0Q is actually a positive approximant. It would suffice to show that each $P_0 - A_\gamma$ and $P_0 - A'_\gamma$ constructed in the proof of Lemma 4.8 is a positive approximant. The arguments given in that earlier proof show that

$$0 \leqq A_\gamma \leqq A_0 \leqq P_0 \quad \text{and} \quad 0 \leqq A'_\gamma \leqq A_0 \leqq P_0$$

for all positive γ sufficiently small. The continuity of the expression

$$X - (\delta^2 - C^2)^{1/2} + iC$$

in X with respect to the operator norm and Lemma 5.1 show that for γ sufficiently small

$$\|A_\gamma - (\delta^2 - C^2)^{1/2} + iC\| \leqq \delta \quad \text{and} \quad \|A'_\gamma - (\delta^2 - C^2)^{1/2} + iC\| \leqq \delta.$$

Thus each $P_0 - A_\gamma$ and each $P_0 - A'_\gamma$ is a positive approximant. The real linear independence of the set

$$\{A_0 Q_i, A_\gamma, A'_\gamma \colon i = 1, \ldots, p \text{ and } k > j\}$$

proves the theorem.

It is immediate from the preceding theorem and Theorem 3.1 that $\mathcal{P}(T)$ and $\mathcal{P}'(T)$ have the same dimension when T is normal. Although it is apparent that $\mathcal{P}(T)$ is contained in $\mathcal{P}'(T)$, we are unable to determine when the two sets must coincide in general. From Theorem 5.2 of [3] the two convex sets coincide if T has a unique positive approximant or if T has a unique positive near-approximant. In the sixth section of this paper we shall show that the two convex sets coincide if either is one dimensional. The difficulty of handling the general case centers around the positive near-approximants which do not commute with A_0.

§ 6. Extreme Points of $\mathcal{P}(T)$

Before we give our main result on extreme points we state the following lemma which is a consequence of well-known results.

6.1. Lemma. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis which simultaneously diagonalizes the commuting positive operators R_1, R_2, R_3 on the Hilbert space H_0. Then the lower bound operator for R_1, R_2, R_3 constructed by Lemma 3.2 is*

$$\operatorname{diag}\{\mu_1, \ldots, \mu_n\}$$

relative to $\{e_1, \ldots, e_n\}$ where μ_j is $\min\{\langle R_i e_j, e_j\rangle \colon i = 1, 2, 3\}$ *for $j = 1, \ldots, n$.*

6.2. Theorem. *Assume that T is normal and that H_0 is finite dimensional. If $\{e_1, \ldots, e_p\}$ is an orthonormal basis which diagonalizes the restrictions of A_0, P_0 and C to H_0 and if Q is the orthogonal projection onto e_k then $P_0 - A_0 Q$ is an extreme point of each of the sets $\mathcal{P}(T)$ and $\mathcal{P}'(T)$.*

Proof. It was indicated in the first paragraph of the proof of Theorem 5.2 that P_0-A_0Q is a positive near-approximant of T because Q commutes with A_0 and $2(\delta^2-C^2)^{1/2}$. As was shown in that proof the commutativity of Q and C implies that P_0-A_0Q is a positive approximant of T. Let $\mathscr{C}$ denote the convex set $\{P_0-P: P\in\mathscr{P}'(T)\}$. Clearly it suffices to show that A_0Q is an extreme point of $\mathscr{C}$.

Take $P\in\mathscr{P}'(T)$ and note that

$$\delta^2 = \|T-P\|^2 = \|(B-P)^2+C^2\|$$

and so $\delta^2\geqq(B-P)^2+C^2$. Since taking square roots is a monotone operator function we have

$$(\delta^2-C^2)^{1/2} \geqq \left((B-P)^2\right)^{1/2} = |B-P|$$

and it is easily seen that $|B-P|\geqq B-P$ and $|B-P|\geqq P-B$. Thus

$$(\delta^2-C^2)^{1/2} \geqq P-B, \quad P_0 = B+(\delta^2-C^2)^{1/2} \geqq P \geqq 0$$

and

$$(\delta^2-C^2)^{1/2} \geqq B-P, \quad 2(\delta^2-C^2)^{1/2} \geqq P_0-P \geqq 0.$$

Thus P_0-P, which we shall denote by A, is dominated by both P_0 and $2(\delta^2-C^2)^{1/2}$. It follows that:

$$\ker A \supset \operatorname{span}\{\ker P_0, \ker(\delta^2-C^2)^{1/2}\}$$

or

$$(AH)^- \subset (P_0H)^- \cap \left((\delta^2-C^2)^{1/2}H\right)^- = H_0.$$

Apply Lemma 6.1 to the derivation of A_1 in the first paragraph of the proof of Theorem 5.2. It follows that every eigenvalue of A_1 is an eigenvalue of one of the operators $\sqrt{2}(\delta-C)^{1/2}$, $\sqrt{2}(\delta+C)^{1/2}$ and $P_0^{1/2}$ with common eigenvectors and so each eigenvalue of A_0 is an eigenvalue of one of the two operators P_0 and $2(\delta^2-C^2)^{1/2}$ with the same eigenvectors. From the preceding paragraph we know that $P\in\mathscr{P}'(T)$ implies that

$$\text{(I)} \qquad AH \subset (AH)^- = \left((P_0-P)H\right)^- \subset H_0$$

and $0\leqq A\leqq P_0$, $A\leqq 2(\delta^2-C^2)^{1/2}$.

Now assume that $A_0Q=\lambda A_2+(1-\lambda)A_3$ with $\lambda\in(0, 1)$ and $A_2, A_3\in\mathscr{C}$. By the preceding paragraph we have

$$\text{(II)} \qquad \langle A_2e_k, e_k\rangle \leqq \langle A_0e_k, e_k\rangle = \langle A_0Qe_k, e_k\rangle, \quad \langle A_3e_k, e_k\rangle \leqq \langle A_0Qe_k, e_k\rangle.$$

For $e_j\neq e_k$ we have

$$0 \leqq \lambda\langle A_2e_j, e_j\rangle+(1-\lambda)\langle A_3e_j, e_j\rangle = \langle(\lambda A_2+(1-\lambda)A_2)e_j, e_j\rangle = \langle A_0Qe_j, e_j\rangle = 0$$

and so $\langle A_2e_j, e_j\rangle=0=\langle A_3e_j, e_j\rangle$ for $e_j\neq e_k$.

From this and (I) it follows that

$$A_j = \langle\cdot, e_k\rangle\mu_{jk}e_k \text{ for } j = 2, 3.$$

From (II) we conclude that

$$\text{(III)} \qquad \mu_{jk} \leqq \langle A_0 Q e_k, e_k \rangle \text{ for } j = 2, 3$$

and if either inequality were strict it would certainly follow that

$$\lambda A_2 + (1-\lambda) A_3 \neq A_0 Q.$$

Hence equality holds in each inequality of (III); it follows that $A_2 = A_0 Q = A_3$. Apparently, $A_0 Q$ is an extreme point of $\mathscr{C}$ and the theorem is proved.

As we noted earlier the preceding theorem gives a characterization of those normal operators T for which the dimension of $\mathscr{P}(T)$ is one. In that circumstance it also gives a very explicit description of both $\mathscr{P}(T)$ and $\mathscr{P}'(T)$.

6.3. Corollary. *Assume that T is normal and that H_0 is a one dimensional subspace. Let f_0 be a unit vector in H_0 and let λ_0 and A_1 be defined by the equations*

$$\lambda_0 = \min \{\langle P_0 f_0, f_0 \rangle, \langle 2(\delta^2 - C^2)^{1/2} f_0, f_0 \rangle\},$$
$$A_1 = \langle \cdot, f_0 \rangle \lambda_0 f_0.$$

Then $\mathscr{P}(T)$ and $\mathscr{P}'(T)$ conicide with the convex hull of P_0 and $P_0 - A_1$; consequently we have

$$\mathscr{P}'(T) = \mathscr{P}(T) = \{P_0 - \lambda A_1 : \lambda \in [0, 1]\}.$$

Proof. In the second paragraph of the proof of the preceding theorem it was shown that P_0 is an absolutely maximal element of $\mathscr{P}'(T)$ — that is $P \in \mathscr{P}'(T)$ implies $P \leqq P_0$. Obviously P_0 has the same property for $\mathscr{P}(T)$ and it easily follows that P_0 is an extreme point of both sets. In view of the integral third paragraph given in the proof of Theorem 6.2 we see that A_1 above is actually the operator A_0 in Theorem 6.2. By that theorem it follows that $P_0 - A_1$ is an extreme point of both of the sets $\mathscr{P}'(T)$ and $\mathscr{P}(T)$. In view of Theorem 3.1 and Theorem 5.2 the real dimension of each of the convex sets $\mathscr{P}'(T)$ and $\mathscr{P}(T)$ is one and geometrically it is clear that both sets must be the convex hull of P_0 and $P_0 - A_1$. For completeness sake we prove this last assertion. The set $\mathscr{C} = \{P_0 - P : P \in \mathscr{P}'(T)\}$ is one dimensional and contains the zero operation; thus it is spanned by any nonzero operator in the set, for example A_1. So $\mathscr{C} \subset \{cA_1 : c \text{ real}\}$. However, $A \in \mathscr{C}$ implies that

$$0 \leqq A \leqq P_0, \quad 0 \leqq A \leqq 2(\delta^2 - C^2)^{1/2}$$

by the argument given in the second paragraph of the proof of Theorem 6.2. It follows that

$$\mathscr{C} \subset \{cA_1 : c \in [0, 1]\}$$

by the construction of A_1. Hence $\mathscr{P}'(T)$ is contained in the set

$$\{(1-\lambda) P_0 + \lambda (P_0 - A_1) = P_0 - \lambda A_1 : \lambda \in [0, 1]\}.$$

Since $\mathscr{P}'(T)$ is convex and both P_0 and $P_0 - A_1$ are positive approximants, it must be that both $\mathscr{P}'(T)$ and $\mathscr{P}(T)$ coincide with the above set.

§ 7. Open questions

For T a normal operator and H_0 a finite dimensional subspace, where H_0 was defined in section three, we constructed a real basis for the convex set $\mathcal{P}'(T)$ and each element of that basis is a positive approximant. This tends to suggest that $\mathcal{P}(T)$ and $\mathcal{P}'(T)$ might coincide. We now show that $\mathcal{P}'(T)$ can properly contain $\mathcal{P}(T)$. Let T be the four dimensional operator defined by the diagonal matrix diag $\{i, -i, 2i, -2i\}$. In this instance $T=iC$ and $P_0=\mathrm{diag}\,\{\sqrt{3}, \sqrt{3}, 0, 0\}$.

Let A be the 4×4 matrix (a_{ij}) with $a_{11}=a_{12}=a_{21}=a_{22}=\sqrt{3}/2$ and all other entries equal to zero. Then P_0-A is a positive near-approximant for T but it is not a positive approximant. Thus $\mathcal{P}(T)$ is properly contained in $\mathcal{P}'(T)$ and this gives rise to our first question.

Question 1. What characterizes those normal operators T for wich $\mathcal{P}(T)=$ $=\mathcal{P}'(T)$?

The procedure for obtaining basis elements of $\mathcal{P}'(T)$ which do not necessarily commute with A_0 is less explicit than the construction of basis elements which do commute with A_0. That observation and the remarks of the preceding paragraph suggest several questions.

Question 2. Assuming that H_0 is finite dimensional, what conditions on T suffice for $\mathcal{P}(T)$ to have only a finite number of extreme points? What suffices for $\mathcal{P}'(T)$ to have only a finite number of extreme points?

Our last question would be considerably more interesting if the preceding question had been answered.

Question 3. What is an extreme point of $\mathcal{P}'(T)$ which fails to commute with A_0?

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Subnormal operators with nontrivial quasinormal extensions

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1. Introduction. PUTNAM in [2] developed some interesting properties of certain completely subnormal operators. The results were presented as a generalization of known results about shift operators. It is not clear in [2], however, how much of a generalization they are and which types of completely subnormal operators they apply to.

This note will show that the completely subnormal operators to which Putnam's Theorem 1 applies are quasinormal. This characterization will considerably simplify the proof of that theorem. We will also get an interesting equivalent form of Putnam's Theorem 2.

Our notation will be that of [2]. Let $[X, Y]=XY-YX$ for bounded linear operators X, Y. Recall that an operator T is quasinormal if $[T, T^*T]=0$. Quasinormal operators are always subnormal.

2. Results. Our first result characterizes those T referred to in Theorem 1 of [2].

Theorem 1. *Let T be a completely subnormal operator on a Hilbert space $\mathfrak{H}$ with minimal normal extension N on $\mathfrak{K}$. Let Q denote the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. Then*

$$Q(N^*N) = (N^*N)Q \tag{1}$$

if and only if T is quasinormal.

Proof. Suppose that $T, N, \mathfrak{H}, \mathfrak{K}$, and Q are defined as in Theorem 1. Then relative to the decomposition $\mathfrak{K}=(\mathfrak{K}\ominus\mathfrak{H})\oplus\mathfrak{H}$ we have

$$N = \begin{bmatrix} A & 0 \\ B & T \end{bmatrix}, \tag{2}$$

and

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \tag{3}$$

A trivial matrix computation shows that (1) holds if and only if $T^*B=0$. But if N is normal, then $[T^*, T]=BB^*$. Hence $T^*[T^*, T]=0$ if (1) holds and $T^*(T^*T)=$

$=(T^*T)T^*$ as desired. Conversely, if $T^*(T^*T)=(T^*T)T^*$, then $T^*BB^*=0$. Hence $T^*B=0$ and (1) holds.

If T is quasinormal, then BROWN has shown [1] that T can be written as

$$T=\begin{bmatrix} 0 & 0 & 0 & . \\ P & 0 & 0 & . \\ 0 & P & 0 & . \\ . & . & . & . \end{bmatrix} \tag{4}$$

where P is a positive operator. But then the block form (2) of N is

$$N=\left[\begin{array}{cccc|cccc} . & . & . & . & & & & \\ . & 0 & 0 & 0 & & & & \\ . & P & 0 & 0 & . & . & & \\ . & 0 & P & 0 & 0 & . & & \\ \hline . & 0 & 0 & P & 0 & 0 & 0 & . \\ . & 0 & 0 & 0 & P & 0 & 0 & . \\ . & 0 & 0 & 0 & 0 & P & 0 & . \\ . & . & . & . & . & . & . & . \end{array}\right] \tag{5}$$

provided T is one to one, which it is if it is completely subnormal.

As an immediate consequence we get that if T is completely subnormal and satisfies (1), then the unitary operator in its polar form *is* a bilateral shift and (1.4) of [2] follows immediately.

Using Theorem 1 and a slight modification of the proof of Theorem 2 in [2] we get:

Theorem 2. *Let T be a completely subnormal operator on $\mathfrak{H}$ with minimal normal extension N on $\mathfrak{K}$. Then either*

(i) *$\mathfrak{K}$ is the least subspace containing $\mathfrak{H}$ and invariant under N and N^*N, or*

(ii) *T has a non-normal quasinormal extension T_1 on $\mathfrak{H}_1 \subseteqq \mathfrak{K}$. That is, there exists a non-trivial invariant subspace $\mathfrak{H}_1$ of N such that $\mathfrak{H} \subseteqq \mathfrak{H}_1$ and N restricted to $\mathfrak{H}_1$ is quasinormal.*

Furthermore, (i) *and* (ii) *cannot both be true for T.*

Proof. Suppose that T is a completely subnormal operator. As in [2] let $\mathfrak{H}_1$ denote the least subspace of $\mathfrak{K}$ containing $\mathfrak{H}$ and invariant under both N and N^*N. If $\mathfrak{H}_1=\mathfrak{K}$, then (i) holds. Suppose that $\mathfrak{H}_1 \neq \mathfrak{K}$. Then by Theorem 1, the T_1 of [2, p. 114] is quasinormal so that (ii) holds.

That (i) and (ii) cannot both hold follows from the fact that if T_1 is quasinormal on $\mathfrak{H}_1$ with minimal normal extension N, then N and N^*N leave $\mathfrak{H}_1$ invariant. This is easily seen by observing that if N is given by (5), then N^*N is diagonal.

In [2], PUTNAM views condition (i) as being, in some sense, the opposite behavior from that exhibited by shifts. Our Theorem 2 shows exactly to what extent this is true. Theorem 2 also characterizes those completely subnormal operators with non-trivial quasinormal extensions.

3. An example. It is possible for a completely subnormal operator T to satisfy condition (ii) of Theorem 2 and not be quasinormal.

Example. Let T_1 be the quasinormal operator defined by the matrix (4) on $\mathfrak{h}=\sum_{i=0}^{\infty}\oplus\mathfrak{h}_i$, $\dim\mathfrak{h}_i=2$, with $P=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$. Let $\mathfrak{M}_0$ be the subspace of $\mathfrak{h}_0$ spanned by $\begin{pmatrix}1\\0\end{pmatrix}$. Then $\mathfrak{M}=\mathfrak{M}_0\oplus\sum_{i=1}^{\infty}\oplus\mathfrak{h}_i$ is an invariant subspace for T_1 of codimension one. Let T be the restriction of T to $\mathfrak{M}$ so that T has the matrix

$$T=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & .\\ 1 & 0 & 0 & 0 & 0 & .\\ 1 & 0 & 0 & 0 & 0 & .\\ 0 & 1 & 1 & 0 & 0 & .\\ 0 & 1 & 0 & 0 & 0 & .\\ 0 & 0 & 0 & 1 & 1 & .\\ . & . & . & . & . & .\end{bmatrix}.$$

Then T is not quasinormal since the (2,1) entry of $T(T^*T)$ is 2 while the (2,1) entry of $(T^*T)T$ is 3.

Similar examples can be constructed by taking a quasinormal T_1 in the form (4) on $\mathfrak{h}=\sum_{i=0}^{\infty}\oplus\mathfrak{h}_i$ and defining $\mathfrak{M}=\sum_{i=0}^{\infty}\oplus\mathfrak{M}_i$ where $\mathfrak{M}_i\subseteqq\mathfrak{h}_i$ and $P\mathfrak{M}_i\subseteqq\mathfrak{M}_{i+1}$.

In order to get T to not be quasinormal it is necessary to have some of the $\mathfrak{M}_i$ not be invariant subspaces for P. Care must be taken to guarantee that the minimal normal extension of T is also a normal extension of T_1. Note that P need not be positive in (4) for (4) to define a quasinormal operator. In fact, one only needs that P itself is quasinormal.

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Note

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M. RAJAGOPALAN has brought to my attention that the results of my paper: Uniformly closed Fourier algebras, *Acta Sci. Math.*, **33** (1972), 211—216, are contained — some explicitly, others implicitly — in his paper: Fourier transform in locally compact groups, *ibidem*, **25** (1964), 86—89.

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Factorization of operators in $\mathscr{C}_\varrho$ classes

By E. DURSZT in Szeged

1. Let T be a linear bounded operator on a Hilbert space $\mathfrak{H}$ and ϱ a positive number. If U is a unitary operator on a Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ and

$$T^n h = \varrho P U^n h \quad \text{for} \quad h\in\mathfrak{H}, \qquad n = 1, 2, \dots,$$

where P (as always in the following) is the orthogonal projection onto $\mathfrak{H}$, then we say that U is a unitary ϱ-dilation of T. $\mathscr{C}_\varrho$ denotes the class of those operators which have a unitary ϱ-dilation.

The study of $\mathscr{C}_\varrho$ classes and unitary ϱ-dilations was initiated by B. SZ.-NAGY and C. FOIAŞ [3] and continued by a number of authors. Recently T. ANDO [1] proved that $T\in\mathscr{C}_2$ if and only if there exists a contraction C on $\mathfrak{H}$ such that

$$T = 2(I-C^*C)^{1/2}C.$$

Moreover, using this factorization, he constructed a unitary 2-dilation of T on $\bigoplus_{n=-\infty}^{\infty}\mathfrak{H}_n$ $(\mathfrak{H}_n=\mathfrak{H})$ by a matrix of operator entries.

Our purpose is to generalize Ando's results for $\varrho>0$, $\varrho\neq 1$. ($\mathscr{C}_1$ is the class of contractions, cf. [$*$], Ch.I). Although we shall not explicitly construct a matrix representation of the unitary ϱ-dilation for $T\in\mathscr{C}_\varrho$, we do construct in Proposition 2 an operator-matrix representation of a contractive ϱ-dilation of T. Since an operator-matrix representation of the unitary 1-dilation of a contraction is well known ([2]; [$*$], Ch.I) it is only a matter of computation to combine the two representations to obtain a matrix representation of a unitary ϱ-dilation of T.

2. Suppose U is a unitary ϱ-dilation of T on $\mathfrak{K}$. Consider the subspaces

$$\mathfrak{L}_0 = \bigvee_{n=0}^{\infty} U^{-n}\mathfrak{H}, \quad \mathfrak{L}_1 = \mathfrak{L}_0\vee U\mathfrak{L}_0$$

and denote by Q the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{L}_0$.

For $n=0, 1, \ldots$ and $h, g \in \mathfrak{H}$ we have

$$(QU(QU-T)h, U^{-n}g) = (UQ(U-T)h, U^{-n}g) = ((U-T)h, U^{-n-1}g) =$$
$$= (U^{n+2}h, g) - (U^{n+1}Th, g) = \frac{1}{\varrho}(T^{n+2}h, g) - \frac{1}{\varrho}(T^{n+1}Th, g) = 0.$$

Hence $QU(QU-T)h$ is orthogonal to $\mathfrak{L}_0$; as on the other hand it is contained in $\mathfrak{L}_0$, we have

(1) $$QU(QU-T)h = 0 \quad \text{for} \quad h \in \mathfrak{H}.$$

Also notice that as $\mathfrak{H} \subset \mathfrak{L}_0$ we have

(2) $$PQUh = PUh = \frac{1}{\varrho} Th \quad \text{for} \quad h \in \mathfrak{H}.$$

Now we can prove:

Lemma. *For an arbitrary $h \in \mathfrak{H}$,*

$$\inf_{g \in \mathfrak{H}} \{\|g - (QU-T)h\|^2 - \|QUg\|^2\} = 0.$$

Proof. Because $\mathfrak{L}_1 = \mathfrak{L}_0 \vee U\mathfrak{H}$, we have

(3) $$\mathfrak{L}_1 = \mathfrak{L}_0 \oplus \overline{(I-Q)U\mathfrak{H}}.$$

For $h \in \mathfrak{H}$ we have $U(QU-T)h \in \mathfrak{L}_1$ and by (1) $U(QU-T)h \perp \mathfrak{L}_0$, consequently

(4) $$U(QU-T)h \in \mathfrak{L}_1 \ominus \mathfrak{L}_0.$$

So, using (4) and (3), we conclude that there exists a sequence $g_n \in \mathfrak{H}$ such that

(5) $$U(QU-T)h = \lim_{n \to \infty} (I-Q)Ug_n.$$

Now, again by (4) and by (1), for $g, h \in \mathfrak{H}$ we have

$$0 \leqq \|(I-Q)Ug - U(QU-T)h\|^2 = \|(I-Q)U(g-(QU-T)h)\|^2 =$$
$$= \|U(g-(QU-T)h)\|^2 - \|QU(g-(QU-T)h)\|^2 = \|g-(QU-T)h\|^2 - \|QUg\|^2.$$

Setting g_n for g, on account of (5) this proves the assertion of the Lemma.

Now we are going to prove our main assertions.

3. Proposition 1. *If $T \in \mathscr{C}_\varrho$ $(\varrho > 0,\ \varrho \neq 1)$ then there exists a contraction C on $\mathfrak{H}$ such that*

$$T = \varrho(I + \varrho(\varrho-2)C^*C)^{-1/2}(I - C^*C)^{1/2}C.$$

Proof. Suppose $T \in \mathscr{C}_\varrho$. Then using the above notations set

$$Z = (PU^*QUP|\mathfrak{H})^{1/2}.$$

Clearly Z is a non-negative contraction and

(6) $$\|Zh\| = \|QUh\| \quad \text{for} \quad h \in \mathfrak{H}.$$

Lemma and (2) imply that

$$0 = \inf_{g \in \mathfrak{H}} \{\|g-(QU-T)h\|^2 - \|QUg\|^2\} =$$

$$= \inf_{g \in \mathfrak{H}} \left\{\|g\|^2 - 2\left(\frac{1}{\varrho}-1\right) \operatorname{Re}(g, Th) + \|QUh\|^2 + \left(1-\frac{2}{\varrho}\right)\|Th\|^2 - \|QUg\|^2\right\}.$$

Using (6) and denoting

(7) $$Y = (I-Z^2)^{1/2}$$

the above equality can be converted to

(8) $$\inf_{g \in \mathfrak{H}} \left\{\|Yg\|^2 - 2\left(\frac{1}{\varrho}-1\right) \operatorname{Re}(g, Th) + \|Zh\|^2 + \left(1-\frac{2}{\varrho}\right)\|Th\|^2\right\} = 0$$

for every $h \in \mathfrak{H}$.

We are going to prove that

(9) $$\|T^* g\| \leqq M\|Yg\|$$

for every $g \in \mathfrak{H}$ with a suitable positive M independent of g. Suppose in the contrary that for every positive M there exists $g_M \in \mathfrak{H}$ such that

$$\|T^* g_M\| > M\|Yg_M\|.$$

Now apply (8) with $T^* g_M \operatorname{sgn}\left(\frac{1}{\varrho}-1\right)$ in place of h. Setting Mg_M for g we get

$$0 \leqq M^2\|Yg_M\|^2 - 2M\left|\frac{1}{\varrho}-1\right|\|T^* g_M\|^2 + \|ZT^* g_M\|^2 + \left(1-\frac{2}{\varrho}\right)\|TT^* g_M\|^2 <$$

$$< \left(1 - 2M\left|\frac{1}{\varrho}-1\right| + \|Z\|^2 + \left|1-\frac{2}{\varrho}\right|\|T\|^2\right)\|T^* g_M\|^2 < 0$$

if M is large enough and this is a contradiction.

(9) guarantees the existence of a bounded linear operator X defined everywhere on $\mathfrak{H}$ such that

(10) $$T^* g = XYg \quad \text{if} \quad g \in \mathfrak{H}, \qquad Xf = 0 \quad \text{if} \quad f \in \mathfrak{H} \ominus Y\mathfrak{H}.$$

Now (8) implies that

$$0 = \inf_{g \in \mathfrak{H}} \left\{\|Yg\|^2 - 2\left(\frac{1}{\varrho}-1\right) \operatorname{Re}(XYg, h) + \|Zh\|^2 + \left(1-\frac{2}{\varrho}\right)\|YX^* h\|^2\right\} =$$

$$= \inf_{g \in \mathfrak{H}} \left\{\|Yg\|^2 - 2\left(\frac{1}{\varrho}-1\right) \operatorname{Re}(Yg, X^* h) + \left(\frac{1}{\varrho}-1\right)^2 \|X^* h\|^2 + \|Zh\|^2 - \right.$$

$$\left. - \frac{1}{\varrho^2}\|X^* h\|^2 + \left(\frac{2}{\varrho}-1\right)\|ZX^* h\|^2\right\}.$$

This means that for every $h\in\mathfrak{H}$

$$\|Zh\|^2-\frac{1}{\varrho^2}\left(\|X^*h\|^2+\varrho(\varrho-2)\|ZX^*h\|^2\right)+\inf_{g\in\mathfrak{H}}\left\{\left\|Yg-\left(\frac{1}{\varrho}-1\right)X^*h\right\|^2\right\}=0. \tag{11}$$

For arbitrary $f\in\mathfrak{H}\ominus Y\mathfrak{H}$ (10) implies that $f\perp X^*\mathfrak{H}$. This shows that $\mathfrak{H}\ominus Y\mathfrak{H}\subset\mathfrak{H}\ominus X^*\mathfrak{H}$ and consequently $\overline{X^*\mathfrak{H}}\subset\overline{Y\mathfrak{H}}$. So we conclude for arbitrary $h\in\mathfrak{H}$ that

$$\inf_{g\in\mathfrak{H}}\left\|Yg-\left(\frac{1}{\varrho}-1\right)X^*h\right\|^2=0.$$

This fact together with (11) imply that

$$\|Zh\|=\frac{1}{\varrho}\left(\|X^*h\|^2+\varrho(\varrho-2)\|ZX^*h\|^2\right)^{1/2}. \tag{12}$$

Since Z is a non-negative contraction we have

$$((I+\varrho(\varrho-2)Z^2)h,h)\geqq\begin{cases}\|h\|^2 & \text{if}\quad \varrho\geqq 2\\ (\varrho-1)^2\|h\|^2 & \text{if}\quad 0<\varrho<2,\end{cases}$$

consequently for $\varrho>0$, $\varrho\neq 1$ there exists the positive, boundedly invertible operator $(I+\varrho(\varrho-2)Z^2)^{1/2}$ and, by (12),

$$\|Zh\|=\frac{1}{\varrho}\left\|(I+\varrho(\varrho-2)Z^2)^{1/2}X^*h\right\|.$$

So there exists an operator W on $\mathfrak{H}$ such that

$$\|WZh\|=\|Zh\|\quad\text{for}\quad h\in\mathfrak{H},\qquad Wf=0\quad\text{for}\quad f\in\mathfrak{H}\ominus Z\mathfrak{H}, \tag{13}$$

and

$$WZh=\frac{1}{\varrho}(I+\varrho(\varrho-2)Z^2)^{1/2}X^*h.$$

Now (10) and the invertibility of $(I+\varrho(\varrho-2)Z^2)^{1/2}$ imply that:

$$T=\varrho Y(I+\varrho(\varrho-2)Z^2)^{-1/2}WZ.$$

Let $C=WZ$, then by (7) and (13) we can conclude

$$T=\varrho(I+\varrho(\varrho-2)C^*C)^{-1/2}(I-C^*C)^{1/2}C.$$

Proposition 2. *Suppose C is a contraction on the Hilbert space $\mathfrak{H}$ and $\varrho>0$, $\varrho\neq 1$. Set*

$$A=(I+\varrho(\varrho-2)C^*C)^{-1/2},\quad B=(I-C^*C)^{1/2},\quad B'=(I-CC^*)^{1/2},\quad T=\varrho ABC,$$

and let V be the linear operator defined on $\mathfrak{H}\oplus\mathfrak{H}$ by the matrix of operator entries

$$\begin{bmatrix} ABC & ABB' \\ (\varrho-1)CAC & (\varrho-1)CAB' \end{bmatrix}.$$

Then: (i) *V is a contraction,*

(ii) $V^2h=VTh$ *for* $h\in\mathfrak{H}$,

(iii) $T^nh=\varrho PV^nh$ *for* $h\in\mathfrak{H}$, *and* $n=1,2,\ldots$,

(iv) $T\in\mathscr{C}_\varrho$.

Proof. Observe that C^*C, A, B commute, $B'C=CB$, $C^*B'=BC^*$, and

$$A^2(B^2+(\varrho-1)^2C^*C)=I.$$

Using these facts an easy computation shows that

$$V^*V=\begin{bmatrix}C^*C & C^*B'\\ B'C & B'^2\end{bmatrix},\quad I-V^*V=\begin{bmatrix}B^2 & -BC^*\\ -CB & CC^*\end{bmatrix}=\begin{bmatrix}B & 0\\ 0 & -C\end{bmatrix}\begin{bmatrix}I & I\\ I & I\end{bmatrix}\begin{bmatrix}B & 0\\ 0 & -C^*\end{bmatrix}\geqq 0,$$

and consequently, $\|V\|\leqq 1$.

The following computation proves (ii): For $h\in\mathfrak{H}$

$$V(V-T)\begin{bmatrix}h\\ 0\end{bmatrix}=V\begin{bmatrix}(1-\varrho)ABCh\\ (\varrho-1)CACh\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

From (ii) we deduce

$$V^nh=VT^{n-1}h\quad\text{for}\quad h\in\mathfrak{H},\quad n=1,2,\ldots. \tag{14}$$

For $n=1$, (iii) is an immediate consequence of the definition of V and T, and the general case then follows using (14).

Now by virtue of (iii), every unitary 1-dilation U_1 of the contraction V is a ϱ-dilation of T. So (iv) is proved.

By virtue of Propositions 1 and 2 we have:

Theorem. *Suppose $\varrho>0$ and $\varrho\neq 1$. An operator T on $\mathfrak{H}$ belongs to $\mathscr{C}_\varrho$ if and only if there exists a contraction C on $\mathfrak{H}$ such that*

$$T=\varrho(I+\varrho(\varrho-2)C^*C)^{-1/2}(I-C^*C)^{1/2}C.$$

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Power-bounded operators with invertible characteristic function

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Abstract: Invertibility of the generalized characteristic function of a power-bounded operator is considered. It is proved that a power-bounded operator whose (not necessarily bounded) characteristic function is invertible and whose spectrum has zero Lebesgue measure is a unitary operator. AMS subject classification number: Primary 47A99.

Key words and phrases: power-bounded, similarity, characteristic function.

In this note we wish to consider invertibility of the generalized characteristic function of a power-bounded operator. It is neither surprising nor difficult to see that when the characteristic function is bounded on the open unit disk, the condition for its invertibility is the same as in the case of a contraction. However without any condition on the boundedness of the characteristic function its invertibility always makes the operator similar to a unitary operator. We prove that if in addition the spectrum of such an operator has Lebesgue measure zero then it is in fact a unitary operator.

We consider an operator T on a (separable) Hilbert space $\mathfrak{H}$ and following the notation in [1] we denote the characteristic function of T by $\Theta_T(\lambda)$. Throughout the course of this paper we assume that $\Theta_T(\lambda)$ is invertible whenever $|\lambda|<1$ and write the analytic function $\Omega_T(\lambda)=\Theta_T(\lambda)^{-1}=\sum_{n=0}^{\infty}\lambda^n\omega_n$ where each ω_n is an operator from $\mathfrak{D}_{T^*}$ to $\mathfrak{D}_T$. If $\sup_{|\lambda|<1}\|\Omega_T(\lambda)\|<\infty$ then it follows that, for almost all t, $\Omega_T(re^{it})$ converges strongly to $\Omega_T(e^{it})$ as $r\to 1$, *cf.* [3], Chapter V. Moreover, for u in $H^2(\mathfrak{D}_{T^*})$, $(\Omega_T u)(\lambda)=\Omega_T(\lambda)u(\lambda)$ is in $H^2(\mathfrak{D}_T)$ and thus Ω_T defines a bounded operator on $H^2(\mathfrak{D}_{T^*})$.

Lemma 1. *If T is power-bounded and $\Theta_T(\lambda)$ is an invertible operator for every λ in the open unit disk, then $\sigma(T)$ is contained in the unit circle.*

Proof. Note that $\Theta_T(0)=-TJ_T|\mathfrak{D}_T$. Since $J_T|\mathfrak{D}_T$ is a symmetry it follows that T maps $\mathfrak{D}_T$ onto $\mathfrak{D}_{T^*}$. If $h\in\mathfrak{D}_{T^*}^{\perp}$ then $h=TT^*h$. Hence T is onto. Since $\Theta_{T^*}(0)=$

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$=\Theta_T^*(0)$ is invertible, the same argument proves that T^* is onto. Thus T is invertible. A simple modification of [3, p. 229, Sec. 3] shows that by taking Möbius transforms it follows that $T-\lambda$ is invertible for all λ in the open unit disk.

We are now able to state the following proposition. Though the fact seems to be known, we have not been able to find an explicit proof of it anywhere. Hence for the sake of completeness we give a brief sketch of the proof.

Proposition 1. *If T is power-bounded and $\sup_{|\lambda|<1}\|\Omega_T(\lambda)\|<\infty$ then T is similar to a unitary operator.*

Proof of this proposition rests on the result stated by Foiaş, in [2, p. 437] which we prove as a lemma.

Lemma 2. *If T is invertible then $\Theta_{T^{-1}}(\lambda^{-1})$ coincides with $\Theta_T(\lambda)$ up to constant affine factors.*

Proof. Note that $I-T^{-1*}T^{-1}=-(TT^*)^{-1}(I-TT^*)=-(I-TT^*)(TT^*)^{-1}=$ $=-(TT^*)^{-1/2}(I-TT^*)(TT^*)^{-1/2}$. Hence $\mathfrak{D}_{T^{-1}}=\mathfrak{D}_{T^*}$ and similarly $\mathfrak{D}_T=\mathfrak{D}_{T^{*-1}}$. Also $T^{-1}\mathfrak{D}_{T^*}=\mathfrak{D}_{T^{-1*}}$ and $T^*\mathfrak{D}_{T^{-1}}=\mathfrak{D}_T$. If $S=(TT^*)^{-1/2}$ then $\|Q_{T^{-1}}h\|^2=$ $=\|Q_{T^*}Sh\|^2$ and hence there exists a unitary operator $Z:\mathfrak{D}_{T^{-1}}\to\mathfrak{D}_{T^*}$ such that $ZQ_{T^{-1}}=Q_{T^*}S=SQ_{T^*}$. Now

$$\Theta_T(\xi)Q_TJ_T|\mathfrak{D}_T = Q_{T^*}(I-\xi T^*)^{-1}(\xi-T)|\mathfrak{D}_T \quad \text{[see 3, p. 227]}$$

and

$$\Theta_{T^{-1}}(\xi^{-1})Q_{T^{-1}}J_{T^{-1}}|\mathfrak{D}_{T^{-1}} = Q_{T^{*-1}}(I-\xi^{-1}T^{*-1})^{-1}(\xi^{-1}-T^{-1})|\mathfrak{D}_{T^{-1}} =$$

$$= Q_{T^{*-1}}T^*(I-\xi T^*)^{-1}(\xi-T)T^{-1}|\mathfrak{D}_{T^{-1}} = T^*Q_{T^{-1}}(I-\xi T^*)^{-1}(\xi-T)T^{-1}|\mathfrak{D}_{T^{-1}} =$$

$$= T^*Z^{-1}SQ_{T^*}(I-\xi T^*)^{-1}(\xi-T)T^{-1}|\mathfrak{D}_{T^{-1}} = T^*Z^{-1}S\Theta_T(\xi)Q_TJ_TT^{-1}|\mathfrak{D}_{T^{-1}} =$$

$$= T^*Z^{-1}S\Theta_T(\xi)Q_TJ_TT^{-1}|\mathfrak{D}_{T^{-1}} = T^*Z^{-1}S\Theta_T(\xi)J_TT^{-1}Q_{T^*}|\mathfrak{D}_{T^{-1}}.$$

Hence

$$\Theta_{T^{-1}}(\xi^{-1})J_{T^{-1}}Z^{-1}SQ_{T^*} = T^*Z^{-1}S\Theta_T(\xi)J_TT^{-1}Q_{T^*}$$

and thus

$$\Theta_{T^{-1}}(\xi^{-1})J_{T^{-1}}Z^{-1}S|\mathfrak{D}_{T^*} = T^*Z^{-1}S\Theta_T(\xi)J_TT^{-1}|\mathfrak{D}_{T^{-1}}$$

which gives the result.

Proof of proposition 1. By lemma 1, $\Theta_T(\xi)$ is defined for all ξ off the unit circle and $\Theta_T^{-1}(\xi)=J_T\Theta_T(\bar{\xi}^{-1})^*J_{T^*}|\mathfrak{D}_{T^*}$ by [1, p. 129]. Hence there exist bounded operators X and Y such that for $|\xi|<1$, $\Theta_{T^{-1}}(\xi)=X\Omega_T(\bar{\xi})^*Y$. Now if $\sup_{|\xi|<1}\|\Omega_T(\xi)\|<\infty$ then by the main theorem in [1, p. 127] T^{-1} is similar to a contraction. The result follows from the well-known theorem of Sz.-Nagy [4].

Lemma 3. $(J_T\Omega_T(\lambda)Q_{T^*}h,\ \Omega_T(\lambda)Q_{T^*}h')=(1-|\lambda|^2)((\lambda-T)^{-1}Q_{T^*}^2J_{T^*}h,\ (\lambda-T)^{-1}Q_{T^*}^2J_{T^*}h')+(Q_{T^*}^2J_{T^*}h,\ h')$ *for all $|\lambda|<1$ and all h, h' in $\mathfrak{H}$.*

Proof. It follows from Lemma 1 that $\Theta_T(\lambda)$ is defined off the unit circle. Hence by [1, p. 129],

$$\Omega_T(\lambda) = [-T^* J_{T^*} + J_T Q_T(\lambda - T)^{-1} Q_{T^*} J_{T^*}] | \mathfrak{D}_{T^*}$$

and

$$\Omega_T(\lambda) Q_{T^*} = J_T Q_T(\lambda - T)^{-1}(I - \lambda T^*).$$

Since $J_T Q_T^2 = Q_T^2 J_T = I - T^* T$ the argument in [3, chap. 6, sec. 4] gives the required relation.

Corollary 1. *For all h in $\mathfrak{H}$,* $(J_T \omega_0 Q_{T^*} h, \omega_0 Q_{T^*} h) = \|h\|^2 - \|T^* h\|^2 + \|T^{-1} Q_{T^*}^2 J_{T^*} h\|^2$.

Corollary 2. *If* $\sup_{|\lambda|<1} \|\Omega_T(\lambda)\| < \infty$ *and* $m(\sigma(T)) = 0$ *then for almost all t and all h, h' in $\mathfrak{H}$ we have*

$$\big(J_T \Omega_T(e^{it}) Q_{T^*} h, \Omega_T(e^{it}) \Omega_{T^*} h'\big) = \big((I - TT^*) h, h'\big).$$

Proof. If $e^{it} \notin \sigma(T)$ then $\|(re^{it} - T)^{-1}\|$ is bounded in a neighborhood of e^{it} and as $r \to 1-$, $(1 - r^2)\|(re^{it} - T)^{-1}\| \to 0$.

Proposition 2. *If* $\sup_{|\lambda|<1} \|\Omega_T(\lambda)\| < \infty$ *and* $m(\sigma(T)) = 0$ *then T is a unitary operator.*

It is convenient to break up the proof in two steps, formulated as Lemma 4 and 5.

Lemma 4. *For all h in $\mathfrak{H}$,* $\sum_{n=1}^{\infty} (J_T \omega_n Q_{T^*} h, \omega_n Q_{T^*} h) = -\|T^{-1} Q_{T^*}^2 J_{T^*} h\|^2$.

Proof. For h in $\mathfrak{H}$, $Q_{T^*} h$ can be considered as a constant function in $H^2(\mathfrak{D}_{T^*})$. If $(Ju)(\lambda) = J_T(u(\lambda))$ for u in $H^2(\mathfrak{D}_T)$ and $|\lambda| < 1$ than $H^2(\mathfrak{D}_T)$ becomes a J-space and we have

$$(J\Omega_T(Q_{T^*} h), \Omega_T(Q_{T^*} h)) = \frac{1}{2\pi} \int_0^{2\pi} (J_T \Omega_T(e^{it}) Q_{T^*} h, \Omega_T(e^{it}) Q_{T^*} h)\, dt = \|h\|^2 - \|T^* h\|^2$$

by Corollary 2. On the other hand,

$$\Omega_T(Q_{T^*} h) = \sum_{n=0}^{\infty} \lambda^n \omega_n Q_{T^*} h \quad \text{and} \quad J\Omega_T(Q_{T^*} h) = \sum_{n=0}^{\infty} \lambda^n J_T \omega_n Q_{T^*} h.$$

Thus

$$\|h\|^2 - \|T^* h\|^2 = \sum_{n=0}^{\infty} (J_T \omega_n Q_{T^*} h, \omega_n Q_{T^*} h).$$

Applying Corollary 1 we get the result.

Lemma 5. *For all h in $\mathfrak{H}$,* $\lim_{n\to\infty} \|T^{-n} T^* h - T^{-(n+1)} h\| = 0$ *and hence* $T^* h = T^{-1} h$.

Proof. It follows as in the proof of Lemma 3 that

$$\omega_0 = -[T^* J_{T^*} + Q_T J_T T^{-1} Q_{T^*} J_{T^*}] | \mathfrak{D}_{T^*}$$

and

$$\omega_n = -Q_T J_T T^{-(n+1)} Q_{T^*} J_{T^*} | \mathfrak{D}_{T^*}$$

for $n \geqq 1$.

Hence for $n \geqq 1$,

$$(J_T \omega_n Q_{T^*} h, \omega_n Q_{T^*} h) = \big((I - T^*T)[T^{-n}T^*h - T^{-(n+1)}h], [T^{-n}T^*h - T^{-(n+1)}h]\big) =$$
$$= \|T^{-n}T^*h - T^{-(n+1)}h\|^2 - \|T^{-(n-1)}T^*h - T^{-n}h\|^2.$$

Thus

$$\sum_{n=1}^{\infty} (J_T \omega_n Q_{T^*} h, \omega_n Q_{T^*} h) = \lim_{m \to \infty} \sum_{n=1}^{m} (J_T \omega_n Q_{T^*} h, \omega_n Q_{T^*} h) =$$
$$= \lim_{m \to \infty} \|T^{-m}T^*h - T^{-(m+1)}h\|^2 - \|T^*h - T^{-1}h\|^2.$$

Since $\|T^{-1}Q_{T^*}^2 J_{T^*} h\|^2 = \|T^{-1}(I - TT^*)h\|^2 = \|T^{-1}h - T^*h\|^2$ an application of Lemma 4 gives that $\lim_{n \to \infty} \|T^{-n}(T^*h - T^{-1}h)\| = 0$. Since T is power-bounded we have $T^*h = T^{-1}h$.

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Strongly reductive operators

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§ 1. Reductive

An operator T on a Hilbert space $\mathfrak{H}$ is said to be *reductive* if each subspace of $\mathfrak{H}$ invariant under T reduces T. (In this paper operators are bounded linear transformations, Hilbert spaces are complex, separable and infinite-dimensional, and subspaces are closed linear manifolds.) Using orthogonal projections instead of subspaces the definition can be expressed algebraically: T is reductive if $PT=TP$ whenever $P^2=P=P^*$ and $(1-P)TP=0$.

All Hermitian operators are reductive, and there are many examples of non-Hermitian reductive operators. However, no non-normal operator has been shown to be reductive, which suggests the following conjecture:

Reductive operator conjecture. Every reductive operator is normal.

It is a remarkable fact that this conjecture is equivalent to the perhaps best known conjecture in operator theory:

Invariant subspace conjecture. Every operator on a Hilbert space has a non-trivial invariant subspace.

(The subspaces $\{0\}$ and $\mathfrak{H}$ are the trivial subspaces of the Hilbert space $\mathfrak{H}$; all other subspaces are non-trivial.)

Theorem 1.1. (DYER, PEDERSON and PORCELLI [3]). *The reductive operator conjecture is true if and only if the invariant subspace conjecture is true.*

The question "Which normal operators are reductive?" has been studied by several authors, beginning with WERMER in 1952. He solved the problem completely for unitary operators, and obtained certain sufficient conditions for arbitrary normal operators.

*) The author gratefully acknowledges the support of a Rothmans Fellowship.

Theorem 1.2. (Wermer [11]). *If the spectrum, $\sum(T)$, of the normal operator T neither divides the plane nor has interior, then T is reductive. ($\sum(T)$ is a compact subset of the complex plane* $\mathbf{C}$; *we say $\sum(T)$ divides the plane if its complement is disconnected.)*

The proof of the theorem depends on a special case of the following well known theorem. The theorem and its proof are included because the theorem is used several times in this paper.

Theorem 1.3. *Let t be an element of a C^*-algebra. Then the following conditions are equivalent:*

(i) *t is normal and the spectrum of t, $\sum(t)$, neither divides the plane nor has interior;*
(ii) *t^* is the limit in norm of a sequence of polynomials in t.*

Proof. Suppose t satisfies either (i) or (ii). Then t is normal and $\mathfrak{C}$, the closed subalgebra generated by 1, t and t^*, is commutative. By the Gelfand—Neumark theorem, the Gelfand mapping $x \to \hat{x}$ of $\mathfrak{C}$ onto $C(\mathfrak{M})$, the algebra of continuous functions on the maximal ideal space $\mathfrak{M}$ of $\mathfrak{C}$, is an isometric isomorphism. Moreover, $\mathfrak{M} = \sum(t)$, and $\hat{t}(z) = z$ for each z in $\sum(t)$. Thus t^* is the limit in norm of a sequence of polynomials in t if and only if the function $\bar{z} \in P(\sum(t))$, the closure in $C(\sum(t))$ of the set of polynomials. The Stone—Weierstrass theorem implies that $\bar{z} \in P(\sum(t))$ if and only if $P(\sum(t)) = C(\sum(t))$, and by Lavrentiev's theorem [5, p. 48] $P(\sum(t)) = C(\sum(t))$ if and only if $\sum(t)$ neither divides the plane nor has interior.

A necessary and sufficient condition for reductivity of a normal operator was obtained by Sarason:

Theorem 1.4. (Sarason [8]). *The normal operator T is reductive if and only T^* is in the closure, with respect to the weak operator topology, of the set of polynomials in T.*

In a subsequent paper he obtains the following spectral criterion for reductivity:

Theorem 1.5. (Sarason [9]). *Let T be a normal operator and let μ be a finite positive measure in the plane which is mutually absolutely continuous with the spectral measure of T. Then T is reductive if and only if the set of polynomials is weak-star dense in $L^\infty(\mu)$.*

In [9] he solves the problem "For which finite positive measures μ are the polynomials weak-star dense in $L^\infty(\mu)$?" so the problem "Which normal operators are reductive?" is solved. However, the solution of the approximation problem for measures is not easy to write down, and is not readily applicable as a test for reductivity of an arbitrary normal operator. The interested reader is referred to the paper.

In this paper we introduce the notion of strong reductivity for operators. We obtain some basic properties of strongly reductive operators, and study the question "Which normal operators are strongly reductive?" We show that the condition on $\sum(T)$ in theorem 1.2, which is sufficient for reductivity of the normal operator T, is both necessary and sufficient for strong reductivity of T.

In § 4 we consider reductivity in the Calkin algebra $\mathfrak{A}$ (definitions will be given) and obtain a necessary and sufficient condition for reductivity in $\mathfrak{A}$ of a normal element in $\mathfrak{A}$. The methods will resemble those used earlier in the paper.

§ 2. Strongly reductive

We say an operator T is strongly reductive if each subspace which is "almost invariant" under T "almost reduces" T. Precisely, the condition is expressed as follows:

Definition 2.1. The operator T is *strongly reductive* if for each $\varepsilon>0$ there is a $\delta>0$ such that $\|PT-TP\|<\varepsilon$ whenever $P^2=P=P^*$ and $\|(1-P)TP\|<\delta$.

Since $\|PT-TP\|=\max\{\|(1-P)TP\|, \|(1-P)T^*P\|\}$, $\|PT-TP\|$ may be replaced in the definition by $\|(1-P)T^*P\|$, and in this alternative form the condition was mentioned by Moore [7] as a natural strengthening of reductivity. The following theorem provides examples and summarizes some basic properties of strongly reductive operators:

Theorem 2.2. (i) *Hermitian operators are strongly reductive*, (ii) *strongly reductive operators are reductive, and* (iii) *the adjoint of a strongly reductive operator is strongly reductive.*

Proof. Parts (i) and (ii) are trivial consequences of the definitions. For any operator T and any projection P, $\|PT^*-T^*P\|=\|QT-TQ\|$ and $\|(1-P)T^*P\| =\|(1-Q)TQ\|$, where $Q=1-P$, so (iii) follows.

The following theorem provides further examples of strongly reductive operators, and, in view of theorem 1.3, it can be regarded as an extension of Theorem 1.2.

Theorem 2.3. *If T^* is the uniform limit of a sequence of polynomials in the operator T, then T is strongly reductive.*

Proof. We first show that if q is any polynomial and if $\varepsilon>0$, then there is a $\delta>0$ such that $\|(1-P)q(T)P\|<\varepsilon$ whenever P is an orthogonal projection and $\|(1-P)TP\|<\delta$. The proof is by induction on the degree of the polynomial. The statement is trivially true if q is a constant polynomial, for then $(1-P)q(T)P=0$. Suppose the statement is true for polynomials of degree k, and suppose q is a polynomial of degree $k+1$. Let $r(z)=(q(z)-q(0))z^{-1}$. Then r is a polynomial of degree k, and

$$\begin{aligned}\|(1-P)q(T)P\| &= \|(1-P)(q(T)-q(0)I)P\| \\ &= \|(1-P)T(1-P+P)r(T)P\| \\ &\leqq \|(1-P)T(1-P)r(T)P\|+\|(1-P)TPr(T)P\| \\ &\leqq \|T\|\|(1-P)r(T)P\|+\|(1-P)TP\|\|r(T)\|.\end{aligned}$$

It follows that the statement is true for this polynomial q, and so by induction it is true for all polynomials.

Now choose $\varepsilon>0$, choose a polynomial q such that $\|T^*-q(T)\|<\varepsilon/2$, and choose $\delta>0$ such that $\|(1-P)q(T)P\|<\varepsilon/2$ whenever $\|(1-P)TP\|<\delta$. For such a P, $\|(1-P)T^*P\|<\|(1-P)q(T)P\|+\varepsilon/2<\varepsilon$. Thus T is strongly reductive.

Corollary 2.4. *If T is normal, and if $\sum(T)$ neither divides the plane nor has interior, then T is strongly reductive.*

Proof. Apply theorem 1.3 and theorem 2.3.

§ 3. Spectrum

We derive certain properties of the spectrum of a strongly reductive operator. We prove that the spectrum neither divides the plane nor has interior, and, except for isolated normal eigenvalues of finite multiplicity, the spectrum equals the essential spectrum.

Let $\mathfrak{B}(\mathfrak{H})$ denote the algebra of all operators on the Hilbert space $\mathfrak{H}$, and let $\mathfrak{A}$ denote the Calkin algebra, i.e. the factor algebra $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$, where $\mathfrak{K}$ is the ideal of compact operators on $\mathfrak{H}$. Let π denote the canonical map from $\mathfrak{B}(\mathfrak{H})$ onto $\mathfrak{A}$: for each operator T, $\pi(T)$ is the coset in $\mathfrak{A}$ containing T. For each operator T we define the following subsets of the plane:

$$\begin{aligned}\textstyle\sum(T) &= \{\lambda:\lambda-T \text{ has no inverse in } \mathfrak{B}(\mathfrak{H})\}, \\ \Pi(T) &= \{\lambda:\lambda-T \text{ has no left inverse in } \mathfrak{B}(\mathfrak{H})\}, \\ \textstyle\sum_{\text{ess}}(T) &= \{\lambda:\pi(\lambda-T) \text{ has no inverse in } \mathfrak{A}\}, \\ \Pi_{\text{ess}}(T) &= \{\lambda:\pi(\lambda-T) \text{ has no left inverse in } \mathfrak{A}\}.\end{aligned}$$

$\sum(T)$, $\Pi(T)$, $\sum_{\text{ess}}(T)$ and $\Pi_{\text{ess}}(T)$ are called the spectrum, the left spectrum or approximate point spectrum, the essential spectrum, and the left essential spectrum of T, respectively. Each is a non-empty compact subset of the plane, and they are related as follows:

(3.1) $\quad \sum(T) = \Pi(T)\cup\Pi(T^*)^- \quad$ and $\quad \sum_{\text{ess}}(T) = \Pi_{\text{ess}}(T)\cup\Pi_{\text{ess}}(T^*)^-$

where $^-$ denotes complex conjugation. The left spectra $\Pi(T)$ and $\Pi_{ess}(T)$ can be characterised in terms of "boundedness below":

Proposition 3.2. (i) $\lambda\in\Pi(T)$ *if and only if there is a sequence of unit vectors* $\{\varphi_n\}$ *such that* $\|(T-\lambda)\varphi_n\|\to 0$ *as* $n\to\infty$, *and* (ii) $\lambda\in\Pi_{ess}(T)$ *if and only if there is an orthogonal sequence of unit vectors* $\{\varphi_n\}$ *such that* $\|(T-\lambda)\varphi_n\|\to 0$ *as* $n\to\infty$.

Proof. See [6, p. 37] and [4].

Lemma 3.3. *If T is strongly reductive, and if* $\{\varphi_n\}$ *is a sequence of unit vectors such that* $\|(T-\lambda)\varphi_n\|\to 0$ *as* $n\to\infty$ *for some* λ, *then* $\|(T^*-\bar\lambda)\varphi_n\|\to 0$ *as* $n\to\infty$.

Proof. Let P_n be the orthogonal projection onto span $\{\varphi_n\}$. Thus $P_n\psi=(\psi,\varphi_n)\varphi_n$ for each vector ψ, and

$$\|(1-P_n)TP_n\| = \|(1-P_n)T\varphi_n\| = \|(1-P_n)(T-\lambda)\varphi_n\| \to 0 \quad \text{as} \quad n\to\infty.$$

Therefore, since T is strongly reductive, $\|(1-P_n)T^*P_n\|\to 0$ as $n\to\infty$. So $\|T^*\varphi_n-P_nT^*\varphi_n\|\to 0$ as $n\to\infty$. Now $P_nT^*\varphi_n=(T^*\varphi_n,\varphi_n)\varphi_n=(\varphi_n,T\varphi_n)\varphi_n$, and $(\varphi_n,T\varphi_n)\to\bar\lambda$ as $n\to\infty$. Thus it follows that $\|(T^*-\bar\lambda)\varphi_n\|\to 0$ as $n\to\infty$.

Corollary 3.4. *If T is strongly reductive, then* $\Sigma(T)=\Pi(T)$ *and* $\Sigma_{ess}(T)=\Pi_{ess}(T)$,

Proof. From proposition 3.2 and lemma 3.3,

$$\Pi(T)=\Pi(T^*)^- \quad \text{and} \quad \Pi_{ess}(T)=\Pi_{ess}(T^*)^-.$$

Now use equations 3.1.

Theorem 3.5. *If T is strongly reductive, then* $\Sigma(T)$ *is the union of* $\Sigma_{ess}(T)$ *and isolated normal eigenvalues of finite multiplicity.*

Proof. By Corollary 3.4 $\Sigma(T)=\Pi(T)$ and $\Sigma_{ess}(T)=\Pi_{ess}(T)$. Clearly $\Pi_{ess}(T)\subset\Pi(T)$. Suppose $\lambda\in\Pi(T)\setminus\Pi_{ess}(T)$. Since $\lambda\notin\Pi_{ess}(T)$, $\ker(T-\lambda)$ is finite-dimensional, and $T-\lambda$ is bounded below on $\ker(T-\lambda)^\perp$. Since $\lambda\in\Pi(T)$, it follows that $\ker(T-\lambda)$ is non-trivial, and thus λ is an eigenvalue of T of finite multiplicity. Since T is strongly reductive, the eigenvalue λ is a normal eigenvalue, i.e. $T\varphi=\lambda\varphi$ implies $T^*\varphi=\bar\lambda\varphi$.

It remains to be shown that λ is an isolated point of $\Pi(T)$. Since $\ker(T-\lambda)$ is invariant under T and T is strongly reductive, $\ker(T-\lambda)$ reduces T. Let T' denote the restriction of T to $\ker(T-\lambda)^\perp$. Since $T'-\lambda$ is bounded below $\lambda\notin\Pi(T')$, and since $\Pi(T')$ is compact there is a $\delta>0$ such that $\mu\notin\Pi(T')$ whenever $|\mu-\lambda|<\delta$. Now $\Pi(T)=\{\lambda\}\cup\Pi(T')$, so λ is isolated in $\Pi(T)$.

Lemma 3.6. *If N is normal, T is strongly reductive and* $\Sigma(N)\subset\Sigma_{ess}(T)$, *then N is reductive.*

Proof. We may suppose that T and N are operators on the same Hilbert space $\mathfrak{H}$. Let P be an orthogonal projection on $\mathfrak{H}$ such that $(1-P)NP=0$. We shall show that $PN-NP=0$. Let $\mathfrak{H}^{(\infty)}$ be the orthogonal direct sum of copies of $\mathfrak{H}$, indexed by the non-negative integers. Define the following operators on $\mathfrak{H}^{(\infty)}$:

$$S = T\oplus N\oplus N\oplus N\oplus \dots, \qquad P_1 = 0\oplus P\oplus P\oplus P\oplus \dots,$$

$$P_2 = 0\oplus 0\oplus P\oplus P\oplus \dots, \qquad P_3 = 0\oplus 0\oplus 0\oplus P\oplus \dots, \text{ and so on.}$$

By proposition 3.2, Lemma 3.3 and Corollary 3.4, T is "strongly normal on $\sum_{\text{ess}}(T)$" in the sense of Stampfli [10], i.e. for each λ in $\sum_{\text{ess}}(T)$ there is an orthonormal sequence of vectors $\{\varphi_n\}$ such that $\|(T-\lambda)\varphi_n\|\to 0$ and $\|(T^*-\bar{\lambda})\varphi_n\|\to 0$ as $n\to\infty$. Thus, by a theorem of STAMPFLI [10], there is an isometric isomorphism W, from $\mathfrak{H}$ to $\mathfrak{H}^{(\infty)}$, and a compact operator K on $\mathfrak{H}^{(\infty)}$ such that

$$WTW^{-1} = S+K.$$

Since T is strongly reductive, so is $S+K$. Now

$$\|(1-P_n)(S+K)P_n\| \leqq \|(1-P_n)SP_n\| + \|(1-P_n)KP_n\| =$$
$$= \|(1-P)NP\| + \|(1-P_n)KP_n\| \leqq \|KP_n\|.$$

Now $P_n\to 0$ strongly as $n\to\infty$, and since K is compact it follows that $\|KP_n\|\to 0$ as $n\to\infty$. Thus $\|(1-P_n)(S+K)P_n\|\to 0$ as $n\to\infty$, and since $S+K$ is strongly reductive it follows that $\|P_n(S+K)-(S+K)P_n\|\to 0$ as $n\to\infty$. Now

$$\|P_n(S+K)-(S+K)P_n\| \geqq \|P_nS-SP_n\| - \|P_nK\| - \|KP_n\| =$$
$$= \|PN-NP\| - \|P_nK\| - \|KP_n\|.$$

As before $\|P_nK\|\to 0$ and $\|KP_n\|\to 0$ as $n\to\infty$, so $PN-NP=0$. Thus N is reductive.

Lemma 3.7. *If X is a compact set in the plane which either divides the plane or has interior, then there is a normal operator N which is not reductive, and whose spectrum is contained in X.*

Proof. Let $\hat{X}$ denote the union of X and all bounded components of the complement of X. Then $\hat{X}$ is compact and, by the hypothesis, $\hat{X}$ has non-empty interior. Let G be a component of the interior of $\hat{X}$, and let λ be a point in G. Let m be the harmonic measure on $\hat{X}$ evaluated at λ [2, p. 77]. The measure m is a probability measure, its support, supp m, is the boundary of G, ∂G, and it is the unique representing measure for the complex homomorphism "evaluation at λ" on the Dirichlet algebra $R(\hat{X})$ [5, chapter II.]. ($R(\hat{X})$ is the closure in $C(\hat{X})$ of the set of all rational functions with poles off $\hat{X}$.) That is, for any function f in $R(\hat{X})$, $f(\lambda)= =\int f(z)\,\mathrm{dm}\,(z)$.

Let N be the normal operator of multiplication by z on the Hilbert space $L^2(m)$. Then $\sum(N)=\operatorname{supp} m=\partial G\subset\partial\hat{X}\subset\partial X\subset X$. Let $H^2(m)$ denote the closure in $L^2(m)$ of the set of polynomials. Clearly $H^2(m)$ is invariant under N. Now the constant function $1\in H^2(m)$, and $((N-\lambda)^*1)(z)=\bar{z}-\bar{\lambda}$. If p is any polynomial then $(p(z),\ \bar{z}-\bar{\lambda})=\int p(z)(z-\lambda)\,\mathrm{dm}\,(z)=0$, so $\bar{z}-\bar{\lambda}\in H^2(m)^\perp$. Furthermore $\|\bar{z}-\bar{\lambda}\|^2= =\int|z-\lambda|^2\ \mathrm{dm}\,(z)\geqq\operatorname{dist}(\lambda,\partial G)^2>0$, so it follows that $H^2(m)$ does not reduce N, and thus N is not reductive.

Theorem 3.8. *If T is strongly reductive, then $\sum(T)$ neither divides the plane nor has interior.*

Proof. In view of Theorem 3.5 it is sufficient to show that $\sum_{\text{ess}}(T)$ neither divides the plane nor has interior; but this follows from Lemma 3.6 and Lemma 3.7.

Corollary 3.9. *Let T be a normal operator. Then the following conditions are equivalent:*

(i) *T is strongly reductive,*

(ii) *$\sum(T)$ neither divides the plane nor has interior,*

(iii) *T^* is the uniform limit of a sequence of polynomials in T.*

Proof. By theorem 3.8 (i) implies (ii), and by Corollary 2.4 (ii) implies (i), so (i) and (ii) are equivalent. By Theorem 1.3 (ii) and (iii) are equivalent.

§ 4. Essentially reductive

In [7] MOORE shows how the concept of reductivity can be extended from operators to elements of an arbitrary C^*-algebra. The idea is simply to use the algebraic formulation of the definition of reductivity: say an element t in a C^*-algebra is reductive if $pt=tp$ whenever $p^2=p=p^*$ and $(1-p)tp=0$. He devotes particular attention to the Calkin algebra $\mathfrak{A}$, and we shall provide the solution to a problem he poses concerning reductivity in $\mathfrak{A}$: "Which normal elements of $\mathfrak{A}$ are reductive elements of $\mathfrak{A}$?" First he shows how the problem can be stated in terms of operators.

Definition 4.1. An operator T is *essentially reductive* if $PT-TP$ is compact whenever $P^2=P=P^*$ and $(1-P)TP$ is compact.

Definition 4.2. An operator T is *essentially normal* if T^*T-TT^* is compact.

Moore's problem in these terms is : "Which essentially normal operators are essentially reductive?" He provides the following partial answers:

Theorem 4.3. (MOORE [7]). *If T is essentially normal and if $\sum_{\text{ess}}(T)$ neither divides the plane nor has interior, then T is essentially reductive.*

The proof is based on theorem 1.3.

He also proves the following theorem which he uses to obtain a partial converse to theorem 4.3:

Theorem 4.4. (MOORE [7]). *If N is normal, T is essentially reductive, and $\sum(N)\subset\sum_{\text{ess}}(T)$, then N is reductive.*

The statement and proof of the theorem are analogous to the statement and proof of Lemma 3.6. Using theorem 4.2 Moore shows that if T is essentially normal and if $\sum_{\text{ess}}(T)$ either has interior, or (more generally) contains a closed analytic Jordan curve, then T is not essentially reductive.

Lemma 3.7 provides a full converse to theorem 4.3 when coupled with Theorem 4.4.

Theorem 4.5. *If T is essentially normal, then the following conditions are equivalent:*

(i) *T is essentially reductive,*

(ii) *$\sum_{\mathrm{ess}}(T)$ neither divides the plane nor has interior,*

(iii) *T^* is the uniform limit of a sequence $\{p_n(T)+K_n\}$, where each p_n is a polynomial and each K_n is compact.*

Proof. By Theorem 4.4 and Lemma 3.7 (i) implies (ii), and by Theorem 4.3 (ii) implies (i), so (i) and (ii) are equivalent. Condition (iii) is precisely the condition under which $\pi(T^*)$ is the limit in norm of a sequence of polynomials in $\pi(T)$, so by Theorem 1.3 (ii) and (iii) are equivalent.

Corollary 4.6. *If T is essentially normal and essentially reductive, then T is a compact perturbation of a strongly reductive, normal operator.*

Proof. By Theorem 4.5 $\sum_{\mathrm{ess}}(T)$ does not divide the plane, so by a theorem of Brown, Douglas and Fiillmore [1, p. 119], T is a compact perturbation of a normal operator N such that $\sum(N)=\sum_{\mathrm{ess}}(T)$. By corollary 2.4 N is strongly reductive.

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Integrability theorems for Jacobi series and Parseval's formulae

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1. Introduction and statement of main results

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n and order (α, β), defined by

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\}, \qquad \alpha, \beta > -1.$$

These plynomials are orthogonal on $(-1, 1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ and normalized by

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}.$$

For convenience we often set $x = \cos\theta$. The functions $P_n^{(\alpha,\beta)}(\cos\theta)$ are orthogonal on $(0, \pi)$ with respect to weight function

$$\varrho^{(\alpha,\beta)}(\theta) = \left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1}$$

and satisfy

$$(1.2)\ \frac{1}{\omega_n^{(\alpha,\beta)}} = \int_0^\pi \{P_n^{(\alpha,\beta)}(\cos\theta)\}^2 \varrho^{(\alpha,\beta)}(\theta)\,d\theta = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$$

(notice $\omega_0^{(-1/2,-1/2)} = 1/\pi$).

A Jacobi series is of the form

$$(1.3) \qquad \sum_{n=0}^\infty a_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta),$$

where a_n $(n=0, 1, 2, \ldots)$ are real numbers. If $f(\theta)$ is Lebesgue-integrable on $(0, \pi)$ with respect to the weight function $\varrho^{(\alpha,\beta)}(\theta)$, then we write $f(\theta) \in L([0, \pi]; \alpha, \beta)$ and denote its Fourier—Jacobi series by

$$(1.4) \qquad f(\theta) \sim \sum_{n=0}^\infty a_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta),$$

where the Fourier—Jacobi coefficients are given by

$$a_n = \int_0^{\pi} f(\theta) P_n^{(\alpha,\beta)}(\cos\theta) \varrho^{(\alpha,\beta)}(\theta)\, d\theta.$$

Definition 1. A function $f(\theta)$ is said to be *R-integrable on* $(0, \pi)$ *with respect to the weight function* $\varrho^{(\alpha,\beta)}(\theta)$ if $f(\theta)$ is Lebesgue-integrable on any closed interval $[\theta_1, \theta_2]$, $0<\theta_1<\theta_2<\pi$, and if

$$\lim_{\substack{\theta_1\to+0\\ \theta_2\to\pi-0}} \int_{\theta_1}^{\theta_2} f(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \int_{\to 0}^{\to\pi} f(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

exists and is finite. Then we write $f(\theta)\in R([0,\pi];\alpha,\beta)$.

Definition 2. A function $\varphi(u)$ is said to be *slowly varying* if $\varphi(u)$ is positive and continuous in $u\geqq 0$, and if

$$\frac{\varphi(tu)}{\varphi(u)} \to 1 \quad \text{as} \quad u\to\infty$$

for every $t>0$.

A slowly varying function $\varphi(u)$ has the following properties.

(S1) $\dfrac{\varphi(tu)}{\varphi(u)} \to 1$ as $u\to\infty$ uniformly for $0<T_1\leqq t\leqq T_2<\infty$,

where T_1 and T_2 are any two fixed values.

(S2) $u^{\gamma}\varphi(u)\to\infty$, $u^{-\gamma}\varphi(u)\to 0$ as $u\to\infty$ for every $\gamma>0$.

(S3) If we set

$$\varphi_1(u) = u^{-\gamma}\sup_{0\leqq v\leqq u}\{v^{\gamma}\varphi(v)\},\quad \varphi_2(u) = u^{\gamma}\sup_{u\leqq v<\infty}\{v^{-\gamma}\varphi(v)\} \quad \text{for}\quad \gamma>0,$$

then $\varphi_m(u)/\varphi(u)\to 1$ as $u\to\infty$ $(m=1,2)$. Furthermore, $u^{\gamma}\varphi_1(u)$ is non-decreasing, and $u^{-\gamma}\varphi_2(u)$ is non-increasing.

(S4) For $\gamma>0$, we have

$$\varphi(tu)\leqq A' t^{-\gamma}\varphi(u) \quad \text{and} \quad \varphi(u/t)\leqq A'' t^{-\gamma}\varphi(u) \quad \text{for every} \quad u\geqq 0,\ 1\geqq t>0,$$

where A' and A'' are positive constants depending only on γ and φ.

(S1), (S2) and (S4) are due to S. IGARI [7], and (S3) is due to S. ALJANČIĆ, R. BOJANIĆ and M. TOMIĆ [1].

C. C. GANSER [6] gave some results with respect to L-integrability of ultraspherical series. First we give a sufficient condition concerning coefficients in order that the Jacobi series (1.3) converges to a function in $0<\theta<\pi$, and then prove five results (Theorems 1, 2 and 3) with respect to R- or L-integrability of the function.

Theorem 1. *Let $\alpha, \beta \geqq -1/2$. Suppose that the Jacobi series* (1.3) *satisfies the conditions:*

(J1) *for* $\alpha > -1/2$,

$$\sum_{n=1}^{\infty} n^{1/2}|\Delta a_n| < \infty \quad \textit{and} \quad n^{1/2}a_n \to 0 \quad \textit{as} \quad n \to \infty \quad (\Delta a_n = a_n - a_{n+1}),$$

(J2) *for* $\alpha = -1/2$,

$$\sum_{n=1}^{\infty} |\Delta(n^{1/2}a_n)| < \infty \quad \textit{and} \quad n^{1/2}a_n \to 0 \quad \textit{as} \quad n \to \infty.$$

Then the Jacobi series (1.3) *converges to a function* $f(\theta)$ *in* $0<\theta<\pi$, *uniformly in* $\varepsilon \leqq \theta \leqq \pi-\varepsilon$ *for every* $0<\varepsilon<\pi$. *Moreover we have:*

(i) *If* $\alpha, \beta \geqq -1/2$, *then*

$$f(\theta)\theta^{\mu-\alpha-1/2}(\pi-\theta)^{\nu-\beta-3/2} \in L([0,\pi];\alpha,\beta) \quad \textit{for any} \quad \mu, \nu > 0.$$

(ii) *If* $\alpha \geqq -1/2$ *and* $\beta > -1/2$, *then*

$$f(\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2} \in R([0,\pi];\alpha,\beta).$$

Remark 1. As it is easily seen from the proof of Theorem 1, we have under the conditions of Theorem 1 that

$$\int_{\to 0}^{\pi} f(\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{\nu-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

exists and is finite if $\alpha, \beta \geqq -1/2$ and $\nu > 0$. The particular case $\alpha=\beta=-1/2$ and $\nu=1$ (Fourier cosine series) is stated by Bary [4; p. 209—211].

Remark 2. Using (3.3) for $\tau=0$, the conditions in (J1) imply $\sum_{n=1}^{\infty} n^{-1/2}|a_n| < \infty$. Hence, by $\Delta(n^{1/2}a_n) = n^{1/2}\Delta a_n + a_{n+1}\Delta n^{1/2}$, the conditions in (J1) imply the conditions in (J2). Conversely, it is clear that the conditions

$$\sum_{n=1}^{\infty} |\Delta(n^{1/2}a_n)| < \infty, \quad \sum_{n=1}^{\infty} n^{-1/2}|a_n| < \infty \quad \text{and} \quad n^{1/2}a_n \to 0 \quad \text{as} \quad n \to \infty$$

imply the conditions in (J1).

Theorem 2. *Let $\alpha \geqq -1/2$ and $\beta > -1/2$. Suppose that the Jacobi series* (1.3) *satisfies the conditions:*

(J3) *for* $\alpha > -1/2$,

$$\sum_{n=1}^{\infty} |\Delta a_n| n^{1/2} \log(n+1) < \infty \quad \textit{and} \quad n^{1/2}a_n \to 0,$$

(J4) *for* $\alpha=-1/2$,

$$\sum_{n=1}^{\infty}|\Delta(n^{1/2}a_n)|\log(n+1)<\infty \quad \textit{and} \quad n^{1/2}a_n\to 0.$$

Then the Jacobi series (1.3) *convergens to a function* $f(\theta)$ *in* $0<\theta<\pi$, *and furthermore,*

$$f(\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\in L([0,\pi];\alpha,\beta).$$

Theorem 3. *Let* $\alpha,\beta\geqq -1/2$ *and* $0<\delta<1$, *and let* $\varphi(u)$ *be a slowly varying function. Suppose that* $\{a_n\}$ $(n=0,1,2,\dots)$ *is a non-negative sequence such that* $\{n^{1/2}a_n\}$ $(n=1,2,\dots)$ *is non-increasing and tends to zero as* $n\to\infty$, *and that* $\sum_{n=1}^{\infty}n^{-\delta+1/2}\varphi(n)a_n$ *converges. Then the Jacobi series* (1.3) *converges to a function* $f(\theta)$ *in* $0<\theta<\pi$. *Moreover,*

(i) *if* $-1/2\leqq\beta<1/2$, $\nu\geqq\delta-1$ *and* $\nu>-\beta-1/2$, *then*

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\nu-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in L([0,\pi];\alpha,\beta);$$

(ii) *if* $\beta\geqq 1/2$, *then*

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-5/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in L([0,\pi];\alpha,\beta).$$

The next theorems are extensions of theorems of M. and S. Izumi [8] on the integrability of trigonometric series.

Theorem 4. *Let* $\alpha,\beta\geqq -1/2$, *and let*

$$f(\theta)\in L([0,\pi];\alpha,\beta),\quad g(\theta)\in L\left([0,\pi];\frac{1}{2}\left(\alpha-\frac{1}{2}\right),\frac{1}{2}\left(\beta-\frac{1}{2}\right)\right).$$

Further let

$$f(\theta)\sim\sum_{n=0}^{\infty}a_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta),\quad g(\theta)\sim\sum_{n=0}^{\infty}b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta).$$

Suppose that $g(\theta)$ *is non-negative on* $(0,\pi)$, *that* $\theta^{\alpha+1/2}g(\theta)$ *is non-increasing on* $(0,\pi/2]$, *and that* $(\pi-\theta)^{\beta+1/2}g(\theta)$ *is non-decreasing on* $(\pi/2,\pi)$. *If the series*

$$\sum_{n=1}^{\infty}n^{1/2}a_n\left(\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta+\int_0^{1/n}\theta^{\beta+1/2}g(\pi-\theta)\,d\theta\right)$$

converges absolutely, then $f(\theta)g(\theta)\in R([0,\pi];\alpha,\beta)$ *and Parseval's formula*

$$\sum_{n=0}^{\infty}a_nb_n\omega_n^{(\alpha,\beta)}=\int_{\to 0}^{\to\pi}f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

holds, where the series on the left hand side converges absolutely.

Corollary 1. *Let $\alpha, \beta \geqq -1/2$ and $0<\delta<1$. Let $f(\theta)\in L([0, \pi]; \alpha, \beta)$ and let its Fourier—Jacobi series be* (1.4). *Suppose that $\varphi(u)$ is a slowly varying function such that $u^\gamma\varphi(u)$ is non-decreasing on $(0, \infty)$ for every $\gamma>0$. If the series $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)a_n$ converges absolutely, then*

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in R([0, \pi]; \alpha, \beta).$$

Theorem 5. *Let $\alpha, \beta \geqq -1/2$. Suppose that the Jacobi series*

$$\sum_{n=0}^{\infty} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) \tag{1.5}$$

satisfies the following conditions:

(J5) $b_0 \geqq 0$,

(J6) $\{n^{1/2}b_n\}$ $(n=1, 2, \ldots)$ *is a non-increasing sequence converging to zero,*

(J7) $\sum_{n=1}^{\infty} n^{-1/2}b_n$ *converges when* $\alpha > -1/2$.

Let

$$G(\theta) = \begin{cases} \sum_{n=0}^{[1/\theta]} (n+1)^{\alpha+1} b_n & \text{for} \quad 0 \leqq \theta \leqq \frac{\pi}{2} \\ \sum_{n=0}^{[1/(\pi-\theta)]} (n+1)^{\beta} b_n & \text{for} \quad \frac{\pi}{2} < \theta \leqq \pi, \end{cases} \tag{1.6}$$

where $[u]$ denotes the largest integer $\leqq u$. Moreover, let $f(\theta)\in L([0, \pi]; \alpha, \beta)$ and let its Fourier—Jacobi series be (1.4). *Then the series* (1.5) *converges to a function $g(\theta)$ in $0<\theta<\pi$. Moreover, if $f(\theta)G(\theta)\in L([0, \pi]; \alpha, \beta)$, then the series $\sum_{n=1}^{\infty} a_n b_n \omega_n^{(\alpha,\beta)}$ converges and Parseval's formula*

$$\int_0^\pi f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \sum_{n=0}^{\infty} a_n b_n \omega_n^{(\alpha,\beta)}$$

holds, the integral converging absolutely.

Theorem 5 has the following two corollaries.

Corollary 2. *Let $\beta \geqq \alpha > -1/2$ and $0<\delta<\alpha+1/2$. Let $f(\theta)\in L([0, \pi]; \alpha, \beta)$ and let its Fourier—Jacobi series be* (1.4). *Suppose that $\varphi(u)$ is a slowly varying function such that $u^{-\gamma}\varphi(u)$ is non-increasing on $(0, \infty)$ for every $\gamma>0$. If*

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-1/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in L([0, \pi]; \alpha, \beta),$$

then the series $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)a_n$ converges.

Corollary 3. *Let* $\alpha \geqq \beta \geqq -1/2$ *and* $\alpha+1/2<\delta<\alpha+3/2$. *Let* $f(\theta) \in L([0, \pi]; \alpha, \beta)$ *and let its Fourier—Jacobi series be* (1.4). *Let* $\varphi(u)$ *be defined as in Corollary 2. If*

$$f(\theta)\theta^{\delta-\alpha-3/2}\varphi\left(\frac{1}{\theta}\right) \in L([0, \pi]; \alpha, \beta),$$

then the series $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)a_n$ *converges.*

Throughout the paper, the letter K, with or without a sufix, denotes a positive constant, not necessarily the same on each appearance.

2. Preliminary estimates for Jacobi polynomials

Using Stirling's formula (see [9; p. 32]),

$$\Gamma(u) = \sqrt{2\pi}u^{u-1/2}e^{-u}\left\{1+\frac{1}{12u}+O\left(\frac{1}{u^2}\right)\right\} \quad \text{as} \quad u \to \infty,$$

we have from (1.1) and (1.2), respectively,

$$P_n^{(\alpha,\beta)}(1) = \frac{n^\alpha}{\Gamma(\alpha+1)}\left\{1+\frac{A}{n}+O\left(\frac{1}{n^2}\right)\right\} \quad \text{as} \quad n \to \infty, \tag{2.1}$$

where A is a constant depending only on α, and

$$\omega_n^{(\alpha,\beta)} = 2n\left\{1+\frac{B}{n}+O\left(\frac{1}{n^2}\right)\right\} \quad \text{as} \quad n \to \infty, \tag{2.2}$$

where B is a constant depending only on α and β. Furthermore, by [11; (4.1.3), (4.5.3), (4.5.4)],

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x), \tag{2.3}$$

$$\sum_{k=0}^{n} \omega_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(1) P_k^{(\alpha,\beta)}(x) = \frac{\alpha+1}{2n+\alpha+\beta+2}\omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) P_n^{(\alpha+1,\beta)}(x), \tag{2.4}$$

$$\left(n+\frac{\alpha+\beta+2}{2}\right)(1-x)P_n^{(\alpha+1,\beta)}(x) = (n+\alpha+1)P_n^{(\alpha,\beta)}(x)-(n+1)P_{n+1}^{(\alpha,\beta)}(x). \tag{2.5}$$

By (2.2), (2.3) and [12; (7.32.5)], we have

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2}O(n^{-1/2}) & \text{for } cn^{-1} \leqq \theta \leqq \pi - cn^{-1} \text{ and } \alpha, \beta \geqq -\frac{1}{2}, \\ O(n^\alpha) & \text{for } 0 \leqq \theta \leqq \frac{\pi}{2} \text{ and } \alpha \geqq -\frac{1}{2}, \\ O(n^\beta) & \text{for } \frac{\pi}{2} < \theta \leqq \pi \text{ and } \beta \geqq -\frac{1}{2} \end{cases} \tag{2.6}$$

as $n\to\infty$, where c is a positive constant depending only on α and β. Also,

(2.7)
$$\sqrt{\omega_n^{(\alpha,\beta)}}\,P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}=$$
$$= D\cdot\cos\left\{\left(n+\frac{\alpha+\beta+1}{2}\right)\theta-\frac{\pi}{2}\left(\alpha+\frac{1}{2}\right)\right\}+(\sin\theta)^{-1}O(n^{-1})$$

for $cn^{-1}\leqq\theta\leqq\pi-cn^{-1}$ and $\alpha,\beta>-1$, as $n\to\infty$,

[11; (8.21.18)], and more exatly,

(2.8)
$$\sqrt{\omega_n^{(\alpha,\beta)}}\,P_n^{(\alpha,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}=$$
$$=(D+D^*n^{-1})\cos\left\{\left(n+\frac{\alpha+\beta+1}{2}\right)\theta-\frac{\pi}{2}\left(\alpha+\frac{1}{2}\right)\right\}+$$
$$+\left(En^{-1}\cot\frac{\theta}{2}+E^*n^{-1}\tan\frac{\theta}{2}\right)\sin\left\{\left(n+\frac{\alpha+\beta+1}{2}\right)\theta-\frac{\pi}{2}\left(\alpha+\frac{1}{2}\right)\right\}+(\sin\theta)^{-2}O(n^{-2})$$

for $cn^{-1}\leqq\theta\leqq\pi-cn^{-1}$ and $\alpha,\beta>-1$, as $n\to\infty$,

[3; p. 585], where D, D^*, E and E^* are constants depending only on α and β. Finally,

(2.9)
$$\int_0^1(1-x)^\mu|P_n^{(\alpha,\beta)}(x)|\,dx=\begin{cases}O(n^{-1/2}\log n) & \text{for } 2\mu=\alpha-\frac{3}{2} \text{ and } \alpha,\beta,\mu>-1,\\ O(n^{-1/2}) & \text{for } 2\mu>\alpha-\frac{3}{2} \text{ and } \alpha,\beta,\mu>-1\end{cases}$$

as $n\to\infty$ [11; (7.34.1)].

3. Proofs of Theorems 1 and 2

For the proof of Theorem 1, we need the following lemma.

Lemma 1. *Let $\alpha,\beta\geqq-1/2$. Suppose that a sequence $\{a_n\}$ of real numbers satisfies the condition*

(3.1)
$$\sum_{n=1}^{\infty}n^{\alpha+1/2}|\Delta d_n|<\infty \quad \textit{and} \quad n^{1/2}a_n\to 0 \quad \textit{as} \quad n\to\infty,$$

where $d_n=a_n/P_n^{(\alpha,\beta)}(1)$. Then condition (3.1) *is equivalent to conditions* (J1) *and* (J2).

Proof. By (2.1), we have

$$\Delta d_n=\Gamma(\alpha+1)\left\{a_nn^{-\alpha}\left(1-\frac{A}{n}+O\left(\frac{1}{n^2}\right)\right)-a_{n+1}(n+1)^{-\alpha}\left(1-\frac{A}{n+1}+O\left(\frac{1}{n^2}\right)\right)\right\}=$$
$$=\Gamma(\alpha+1)\left\{\left(1-\frac{A}{n}\right)\Delta(n^{-\alpha}a_n)+a_n\cdot O(n^{-\alpha-2})-a_{n+1}\cdot O((n+1)^{-\alpha-2})\right\}$$

as $n\to\infty$. Since $n^{1/2}a_n\to 0$ as $n\to\infty$, we have

$$\sum_{n=1}^{\infty}(|a_n|n^{-\alpha-2})n^{\alpha+1/2}=\sum_{n=1}^{\infty}n^{-2}\cdot n^{1/2}|a_n|<\infty.$$

Hence

$$\left|\sum_{m=1}^{n}m^{\alpha+1/2}|\Delta d_m|-\Gamma(\alpha+1)\sum_{m=1}^{n}m^{\alpha+1/2}\left|\left(1-\frac{A}{m}\right)\Delta(m^{-\alpha}a_m)\right|\right|\leqq$$

$$\leqq\sum_{m=1}^{n}m^{\alpha+1/2}\left|\Delta d_m-\Gamma(\alpha+1)m^{\alpha+1/2}\left(1-\frac{A}{m}\right)\Delta(m^{-\alpha}a_m)\right|\leqq$$

$$\leqq K\sum_{m=1}^{\infty}m^{\alpha+1/2}(|a_m|m^{-\alpha-2})<\infty.$$

Thus (3.1) is clearly equivalent to the condition

$$\sum_{n=1}^{\infty}n^{\alpha+1/2}|\Delta(n^{-\alpha}a_n)|<\infty \quad\text{and}\quad n^{1/2}a_n\to 0 \quad\text{as}\quad n\to\infty. \tag{3.2}$$

Now, since (J2) coincides with (3.2) for $\alpha=-1/2$, it is enough to show that (J1) is equivalent to (3.2) for $\alpha>-1/2$.

Let $\tau>-1/2$. By Abel's transformation, we have, for $n=2,3,\dots,$

$$\begin{aligned}\sum_{m=1}^{n}m^{-1/2}|a_m|&=\sum_{m=1}^{n-1}\left(\sum_{s=1}^{m}s^{\tau-1/2}\right)\Delta(m^{-\tau}|a_m|)+\left(\sum_{s=1}^{n}s^{\tau-1/2}\right)n^{-\tau}|a_n|\leqq\\&\leqq K_1\sum_{m=1}^{n-1}m^{\tau+1/2}|\Delta(m^{-\tau}a_m)|+K_2n^{1/2}|a_n|.\end{aligned}\tag{3.3}$$

If we assume (J1) or (3.2) for $\alpha>-1/2$, then we get $\sum_{n=1}^{\infty}n^{-1/2}|a_n|<\infty$, using (3.3) for $\tau=0$ or α. By $\Delta(n^{-\alpha}a_n)=n^{-\alpha}\Delta a_n+a_{n+1}\Delta n^{\alpha}$, we get

$$\left|\sum_{m=1}^{n}m^{\alpha+1/2}|\Delta(m^{-\alpha}a_m)|-\sum_{m=1}^{n}m^{1/2}|\Delta a_m|\right|\leqq\sum_{m=1}^{n}|m^{\alpha+1/2}\Delta(m^{-\alpha}a_m)-m^{1/2}\Delta a_m|\leqq$$

$$\leqq\sum_{m=1}^{n}m^{\alpha+1/2}|a_{m+1}\Delta m^{-\alpha}|\leqq K\sum_{m=1}^{n}m^{-1/2}|a_m|.$$

Hence, (J1) is clearly equivalent to (3.2) for $\alpha>-1/2$. Thus Lemma 1 is proved.

Proof of Theorem 1. Let $d_n=a_n/P_n^{(\alpha,\beta)}(1)$. By Lemma 1, conditions (J1) and (J2) are equivalent to (3.1). We put $0<\theta<\pi$. From (2.4) and Abel's transformation, we have, for any m, n $(n>m\geqq 0)$,

$$\sum_{s=m+1}^{n} a_s \omega_s^{(\alpha,\beta)} P_s^{(\alpha,\beta)}(\cos\theta) =$$

$$(3.4)\quad \begin{aligned} &= \sum_{s=m+1}^{n} (\Delta d_s) \frac{\alpha+1}{2s+\alpha+\beta+2} \omega_s^{(\alpha+1,\beta)} P_s^{(\alpha+1,\beta)}(1) P_s^{(\alpha+1,\beta)}(\cos\theta) - \\ &\quad - d_{m+1}\cdot \frac{\alpha+1}{2m+\alpha+\beta+2} \omega_m^{(\alpha+1,\beta)} P_m^{(\alpha+1,\beta)}(1) P_m^{(\alpha+1,\beta)}(\cos\theta) + \\ &\quad + d_{n+1}\cdot \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) P_n^{(\alpha+1,\beta)}(\cos\theta) = \sum_{s=m+1}^{n} I_s - J_1 + J_2, \end{aligned}$$

say. Let N_θ be the smallest positive integer n such that $cn^{-1}\leqq\theta\leqq\pi-cn^{-1}$, where c is a positive constant in (2.6). By (2.1), (2.2) and (2.6), we get, for $n>m\geqq N_\theta$,

$$\sum_{s=m+1}^{n} |I_s| \leqq K_1 \theta^{-\alpha-3/2}(\pi-\theta)^{-\beta-1/2} \sum_{s=m+1}^{n} s^{\alpha+1/2}|\Delta d_s|,$$

$$|J_1| \leqq K_2 \theta^{-\alpha-3/2}(\pi-\theta)^{-\beta-1/2} m^{1/2}|a_m|,$$

$$|J_2| \leqq K_3 \theta^{-\alpha-3/2}(\pi-\theta)^{-\beta-1/2} n^{1/2}|a_n|.$$

Hence, from (3.1) and (3.4), we have, for $n>m\geqq N_\theta$,

$$(3.5)\quad \begin{aligned} &\left|\sum_{s=m+1}^{n} a_s \omega_s^{(\alpha,\beta)} P_s^{(\alpha,\beta)}(\cos\theta)\right| \leqq \sum_{s=m+1}^{n} |I_s| + |J_1| + |J_2| \leqq \\ &\leqq K\theta^{-\alpha-3/2}(\pi-\theta)^{-\beta-1/2}\left(\sum_{s=m+1}^{n} s^{\alpha+1/2}|\Delta d_s| + m^{1/2}|a_m| + n^{1/2}|a_n|\right) \to 0 \quad \text{as} \quad n, m \to \infty. \end{aligned}$$

Hence the Jacobi series (1.3) converges to a function $f(\theta)$ in $0<\theta<\pi$, and furthermore we get

$$(3.6)\qquad f(\theta) = \sum_{n=0}^{\infty} (\Delta d_n) \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) P_n^{(\alpha+1,\beta)}(\cos\theta),$$

where the series converges absolutely in $0<\theta<\pi$ $\left(\omega_0^{(\alpha,\beta)} = (\alpha+1)\omega_0^{(\alpha+1,\beta)}/(\alpha+\beta+2)\right.$ and $\left.P_0^{(\alpha,\beta)}(x) = P_0^{(\alpha+1,\beta)}(x) = 1\right)$.

Next, let $\varepsilon\leqq\theta\leqq\pi-\varepsilon$ for every $0<\varepsilon<\pi$. We put $N_\theta = N_\varepsilon$ for $\theta=\varepsilon$. Then we can replace (3.5) by

$$\left|\sum_{s=m+1}^{n} a_s \omega_s^{(\alpha,\beta)} P_s^{(\alpha,\beta)}(\cos\theta)\right| \leqq K\varepsilon^{-\alpha-\beta-2}\left(\sum_{s=m+1}^{n} s^{\alpha+1/2}|\Delta d_s| + m^{1/2}|a_m| + n^{1/2}|a_n|\right)$$

$$\text{for} \quad n > m \geqq N_\varepsilon.$$

Hence (1.3) converges uniformly in $\varepsilon\leqq\theta\leqq\pi-\varepsilon$ to $f(\theta)$.

We prove (i). By $\alpha, \beta \geqq -1/2$ and $\mu, \nu > 0$, we get $\mu+\alpha+1/2>0$, $\mu+\alpha+1/2>$ $>\alpha-3/2$, $\nu+\beta-1/2>-1$ and $\nu+\beta-1/2>\beta-3/2$. Hence, from (3.6), (2.1), (2.2), (2.3), (2.9) and (3.1), we have

$$\int_0^\pi |f(\theta)|\theta^{\mu-\alpha-1/2}(\pi-\theta)^{\nu-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq$$

$$\leqq K_1 \int_0^\pi |f(\theta)| \left(\sin\frac{\theta}{2}\right)^{\mu+\alpha+1/2} \left(\cos\frac{\theta}{2}\right)^{\nu+\beta-1/2} d\theta \leqq$$

$$\leqq K_2 \sum_{n=0}^{\infty} |\Delta d_n| \frac{\alpha+1}{2n+\alpha+\beta+2}\, \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) \times$$

$$\times \int_0^\pi \left(\sin\frac{\theta}{2}\right)^{\mu+\alpha+1/2} \left(\cos\frac{\theta}{2}\right)^{\nu+\beta-1/2} |P_n^{(\alpha+1,\beta)}(\cos\theta)|\,d\theta \leqq$$

$$\leqq K_3|\Delta d_0| \int_{-1}^{1} (1-x)^{(\mu+\alpha-1/2)/2}(1+x)^{(\nu+\beta-3/2)/2}\,dx +$$

$$+K_4 \sum_{n=1}^{\infty} n^{\alpha+1}|\Delta d_n| \int_{-1}^{1} (1-x)^{(\mu+\alpha-1/2)/2}(1+x)^{(\nu+\beta-3/2)/2}|P_n^{(\alpha+1,\beta)}(x)|\,dx \leqq$$

$$\leqq K_5 + K_4 \sum_{n=1}^{\infty} n^{\alpha+1}|\Delta d_n| \left\{K_6 \int_0^1 (1-x)^{(\mu+\alpha-1/2)/2}|P_n^{(\alpha+1,\beta)}(x)|\,dx +\right.$$

$$\left. +K_7 \int_0^1 (1-x)^{(\nu+\beta-3/2)/2}|P_n^{(\beta,\alpha+1)}(x)|\,dx\right\} \leqq K_5 + K_7 \sum_{n=1}^{\infty} n^{\alpha+1/2}|\Delta d_n| < \infty.$$

Thus (i) is proved.

Finally, we prove (ii). From (3.6) we have, for any $0<\eta,\ \eta'<\pi/2$,

$$\int_\eta^{\pi-\eta'} f(\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\,d\theta =$$

$$= \sum_{n=0}^{\infty} (\Delta b_n) \frac{\alpha+1}{2n+\alpha+\beta+2}\, \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) \times \tag{3.7}$$

$$\times \int_\eta^{\pi-\eta'} P_{n'}^{(\alpha+1,\beta)}(\cos\theta)\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\,d\theta.$$

Now, by the second mean value theorem for integrals, we get

$$\int_{\eta}^{\pi-\eta'} P_n^{(\alpha+1,\beta)}(\cos\theta)\,\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-3/2}\varrho^{(\alpha,\beta)}(\theta)\,d\theta =$$
$$= 2^{-\alpha-\beta-2}\int_{\eta}^{\pi-\eta'} P_n^{(\alpha+1,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2}\times$$
$$(3.8)\qquad \times\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^{\alpha+1/2}\left(\frac{\cos\frac{\theta}{2}}{\frac{\pi-\theta}{2}}\right)^{\beta+3/2} d\theta = 2^{-\alpha-\beta-2}\left(\frac{\sin\frac{\eta}{2}}{\frac{\eta}{2}}\right)^{\alpha+1/2}\left(\frac{\cos\frac{\pi-\eta'}{2}}{\frac{\eta'}{2}}\right)^{\beta+3/2}\times$$
$$\times\int_{\psi}^{\pi-\psi'} P_n^{(\alpha+1,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta,$$

where we may assume $\eta\leqq\psi\leqq\pi/2$ and $\eta'\leqq\psi'\leqq\pi/2$ without loss of generality. We put

$$(3.9)\qquad \int_{\psi}^{\pi-\psi'} P_n^{(\alpha+1,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta = \int_{\psi}^{\pi/2}+\int_{\pi/2}^{\pi-\psi'} = Q_1+Q_2,$$

say. First we estimate Q_1. We consider the case $\psi\leqq cn^{-1}$, where c is a positive constant in (2.6). We put

$$\sqrt{\omega_n^{(\alpha+1,\beta)}}\,Q_1 = \left(\int_{\psi}^{c/n}+\int_{c/n}^{\pi/2}\right)\sqrt{\omega_n^{(\alpha+1,\beta)}}\,P_n^{(\alpha+1,\beta)}(\cos\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta =$$
$$(3.10)\qquad = Q_{1,1}+Q_{1,2}.$$

Then, from (2.1), (2.2) and (2.6), we have

$$(3.11)\qquad |Q_{1,1}|\leqq K\int_0^{c/n} n^{1/2}n^{\alpha+1}\theta^{\alpha+1/2}\,d\theta\leqq K_1\qquad(\alpha\geqq-1/2).$$

Let $\lambda=(\alpha+\beta+2)/2$ and $\zeta=\alpha+3/2$. From (2.7), we get, with $\frac{c}{n}\leqq\sigma\leqq\frac{\pi}{2}$,

$$Q_{1,2} = 2D\int_{c/n}^{\pi/2}\frac{\cos\left\{(n+\lambda)\theta-\frac{\pi\zeta}{2}\right\}}{\sin\theta}\,d\theta+O(n^{-1})\int_{c/n}^{\pi/2}(\sin\theta)^{-2}\,d\theta =$$
$$(3.12)\qquad = 2D\left(\sin\frac{c}{n}\right)^{-1}\int_{c/n}^{\sigma}\cos\left\{(n+\lambda)\theta-\frac{\pi\zeta}{2}\right\}d\theta+O(n^{-1})\int_{c/n}^{\pi/2}\theta^{-2}\,d\theta = O(1)$$
$$\text{as}\quad n\to\infty.$$

Hence, from (3.10), (3.11) and (3.12), we have

$$(3.13)\qquad \sqrt{\omega_n^{(\alpha+1,\beta)}}\,|Q_1|\leqq K_2\quad\text{for}\quad\psi\leqq cn^{-1}.$$

Moreover, by the same method as in (3.12) (replace cn^{-1} with ψ), we get

(3.14) $$\sqrt{\omega_n^{(\alpha+1,\beta)}}|Q_1| \leqq K_3 \quad \text{for} \quad cn^{-1} \leqq \psi.$$

Thus (3.10). (3.13) and (3.14), we have

(3.15) $$\sqrt{\omega_n^{(\alpha+1,\beta)}}|Q_1| \leqq K_4, \quad \text{where} \quad \eta \leqq \psi \leqq \pi/2.$$

Similarly we get

(3.16) $$\sqrt{\omega_n^{(\alpha+1,\beta)}}|Q_2| \leqq K_5, \quad \text{where} \quad \eta' \leqq \psi' \leqq \pi/2.$$

On this occasion, we should notice in particular that, for $\psi' < cn^{-1}$,

$$\left| \int_{\pi-c/n}^{\pi-\psi'} \sqrt{\omega_n^{(\alpha+1,\beta)}}\, P_n^{(\alpha+1,\beta)}(\cos\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+1/2} \left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta \right| \leqq$$

$$\leqq K_6 \int_{\pi-c/n}^{\pi} n^{1/2} n^{\beta} (\pi-\theta)^{\beta-1/2}\, d\theta \leqq K_7 \qquad \left(\beta > -\frac{1}{2}\right)$$

by (2.1) and (2.6). Now, from (3.9), (3.15) and (3.16), we have

(3.17) $$\sqrt{\omega_n^{(\alpha+1,\beta)}} \left| \int_{\psi}^{\pi-\psi'} P_n^{(\alpha+1,\beta)}(\cos\theta) \left(\sin\frac{\theta}{2}\right)^{\alpha+1/2} \left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta \right| \leqq K_8.$$

Thus, by (3.1), (2.1), (2.2), (3.8) and (3.17), we get

(3.18) $$\sum_{n=0}^{\infty} \left| (\Delta b_n) \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) \times \right.$$
$$\left. \times \int_{\eta}^{\pi-\eta'} P_n^{(\alpha+1,\beta)}(\cos\theta)\, \theta^{-\alpha-1/2} (\pi-\theta)^{-\beta-3/2} \varrho^{(\alpha,\beta)}(\theta)\, d\theta \right| \leqq K_9 \sum_{n=0}^{\infty} n^{\alpha+1/2} |\Delta b_n| < \infty.$$

Hence, when we let $\eta, \eta' \to 0$, we have, from (3.7) and (3.18),

$$\int_{\to 0}^{\to \pi} f(\theta)\, \theta^{-\alpha-1/2} (\pi-\theta)^{-\beta-3/2} \varrho^{(\alpha,\beta)}(\theta)\, d\theta =$$

$$= \sum_{n=0}^{\infty} (\Delta b_n) \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) \times$$

$$\times \int_0^{\pi} P_n^{(\alpha+1,\beta)}(\cos\theta)\, \theta^{-\alpha-1/2} (\pi-\theta)^{-\beta-3/2} \varrho^{(\alpha,\beta)}(\theta)\, d\theta,$$

where the series converges absolutely. Thus (ii) is proved.

The proof of Theorem 1 is now complete.

For the proof of Theorem 2, we require the following lemma.

Lemma 2. *Let $\alpha \geqq -1/2$ and $\beta > -1/2$. Suppose that a sequence $\{a_n\}$ of real numbers satisfies the condition*

$$(3.19) \qquad \sum_{n=1}^{\infty} |\Delta \mathrm{d}_n| n^{\alpha+1/2} \log(n+1) < \infty \quad \text{and} \quad n^{1/2} a_n \to 0 \quad \text{as} \quad n \to \infty,$$

where $d_n = a_n / P_n^{(\alpha,\beta)}(1)$. Then condition (3.19) is equivalent to conditions (J3) and (J4).

Proof. If we assume (3.19), then we have for $\alpha \geqq -\frac{1}{2}$, by (2.1),

$$|a_n| n^{1/2} \log(n+1) = P_n^{(\alpha,\beta)}(1) n^{1/2} \log(n+1) \sum_{m=n}^{\infty} \Delta |\mathrm{d}_m| \leqq$$

$$\leqq K \sum_{m=n}^{\infty} |\Delta d_m| m^{\alpha+1/2} \log(m+1) \to 0 \quad \text{as} \quad n \to \infty.$$

Also, if we assume (J3) of (J4), then we get similarly

$$|a_n| n^{1/2} \log(u+1) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus the proof of Lemma 2 is similar to that of Lemma 1.

Proof of Theorem 2. Conditions (J3) and (J4) are included in conditions (J1) and (J2), respectively. Hence the Jacobi series (1.3) converges to a function $f(\theta)$ in $0 < \theta < \pi$, and has the form (3.6). Now, from (2.1), (2.2), (2.9) and Lemma 2, we get for $\alpha \geqq -1/2$, $\beta \geqq -1/2$

$$\int_0^{\pi} |f(\theta)| \theta^{-\alpha-1/2} (\pi-\theta)^{-\beta-3/2} \varrho^{(\alpha,\beta)}(\theta)\, d\theta \leqq$$

$$\leqq \sum_{n=0}^{\infty} |\Delta d_n| \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) \times$$

$$\times \int_0^{\pi} |P_n^{(\alpha+1,\beta)}(\cos\theta)| \theta^{-\alpha-1/2} (\pi-\theta)^{-\beta-3/2} \varrho^{(\alpha,\beta)}(\theta)\, d\theta \leqq$$

$$\leqq K_1 |\Delta d_0| \int_{-1}^{1} (1-x)^{(\alpha-1/2)/2} (1+x)^{(\beta-3/2)/2}\, dx +$$

$$+ K_2 \sum_{n=1}^{\infty} |\Delta d_n| n^{\alpha+1} \int_{-1}^{1} (1-x)^{(\alpha-1/2)/2} (1+x)^{(\beta-3/2)/2} |P_n^{(\alpha+1,\beta)}(x)|\, dx \leqq$$

$$\leqq K_3 + K_4 \sum_{n=1}^{\infty} |\Delta d_n| n^{\alpha+1} \Big\{ K_5 \int_0^1 (1-x)^{(\alpha-1/2)/2} |P_n^{(\alpha+1,\beta)}(x)|\, dx +$$

$$+ K_6 \int_0^1 (1-x)^{(\beta-3/2)/2} |P_n^{(\beta,\alpha+1)}(x)|\, dx \Big\} \leqq K_3 + K_7 \sum_{n=1}^{\infty} |\Delta d_n| n^{\alpha+1/2} \log(n+1) < \infty.$$

Thus Theorem 2 is proved.

4. Proof of Theorem 3

For the proof of Theorem 3, we need the following lemma, due to S. ALJANČIĆ, R. BOJANIĆ and M. TOMIĆ [2].

Lemma 3. *Let $\mu>0$, and let $\varphi(u)$ be a slowly varying function. Suppose that $\{q_n\}$ is a non-increasing sequence tending to zero as $n\to\infty$. Then $\sum_{n=1}^{\infty} n^{\mu-1}\varphi(n)\,q_n$ converges if and only if $\sum_{n=1}^{\infty} n^{\mu}\varphi(u)\Delta q_n$ converges.*

Proof of Theorem 3. Since $\{n^{1/2}a_n\}$ $(n=1, 2, \ldots)$ is non-increasing and tends to zero as $n\to\infty$, we have $\sum_{n=1}^{\infty}|\Delta(n^{1/2}a_n)|<\infty$. Moreover, since $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)|a_n|<<\infty$, we get $\sum_{n=1}^{\infty} n^{-1/2}|a_n|<\infty$ by (S2). Hence, from Remark 2, the sequence $\{a_n\}$ satisfies condition (J1). It is clear that $\{a_n\}$ satisfies condition (J2). Thus, by Theorem 1, the Jacobi series (1.3) converges to a function $f(\theta)$ in $0<\theta<\pi$, and has the form (3.6) (we put $d_n=a_n/P_n^{(\alpha,\beta)}(1)$).

First we shall prove (i). Hence we get, from (2.1) and (2.2),

$$\int_0^{\pi} |f(\theta)|\,\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\nu-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq$$

$$\leqq \sum_{n=0}^{\infty}|\Delta d_n|\frac{\alpha+1}{2n+\alpha+\beta+2}\,\omega_n^{(\alpha+1,\beta)}P_n^{(\alpha+1,\beta)}(1)\int_0^{\pi}\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\nu-\beta-3/2}\times$$

$$\times\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)|P_n^{(\alpha+1,\beta)}(\cos\theta)|\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq \tag{4.1}$$

$$\leqq K\sum_{n=0}^{\infty}|\Delta d_n|\,n^{\alpha+1}\left(\int_0^{\pi/2}+\int_{\pi/2}^{\pi}\right)\theta^{\delta+\alpha-1/2}(\pi-\theta)^{\nu+\beta-1/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)|P_n^{(\alpha+1,\beta)}(\cos\theta)|\,d\theta =$$

$$= K\sum_{n=0}^{\infty}|\Delta d_n|\,n^{\alpha+1}(U_n+V_n),$$

say. Since

$$\Delta d_n=\frac{\Delta(n^{1/2}a_n)}{n^{1/2}P_n^{(\alpha,\beta)}(1)}+\frac{(n+1)^{1/2}a_{n+1}}{n^{1/2}P_n^{(\alpha,\beta)}(1)}\left(1-\frac{n^{1/2}P_n^{(\alpha,\beta)}(1)}{(n+1)^{1/2}P_{n+1}^{(\alpha,\beta)}(1)}\right)$$

and by (1.1),

$$1-\frac{n^{1/2}P_n^{(\alpha,\beta)}(1)}{(n+1)^{1/2}P_{n+1}^{(\alpha,\beta)}(1)}=\frac{\alpha+\dfrac{1}{2}}{n}+O(n^{-2}),$$

and since $\{n^{1/2}a_n\}$ $(n=1, 2, \ldots)$ is non-increasing, there is a positive integer N such that $\Delta d_n \geqq 0$ for all n, $n \geqq N$. Thus, from (4.1), we get

$$\int_0^\pi |f(\theta)|\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\nu-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq \tag{4.2}$$
$$\leqq K_1 + K\sum_{n=N}^{\infty} n^{\alpha+1}(U_n+V_n)\Delta d_n .$$

By (S4), (2.6) and (S2), we have, for $n \geqq N$,

$$U_n \leqq K_2\left(n^{\alpha+1}\int_0^{c/n}\theta^{\delta+\alpha-1/2}\varphi\left(\frac{1}{\theta}\right)d\theta + n^{-1/2}\int_{c/n}^{\pi/2}\theta^{\delta+\alpha-1/2}\varphi\left(\frac{1}{\theta}\right)\cdot\theta^{-\alpha-3/2}\,d\theta\right) \leqq \tag{4.3}$$
$$\leqq K_3 n^{-\delta+1/2}\varphi\left(\frac{n}{c}\right) \leqq K_4 n^{-\delta+1/2}\varphi(n)$$

and

$$V_n \leqq K_4\left(\int_{\pi/2}^{\pi-c/n}(\pi-\theta)^{\nu+\beta-1/2}\varphi\left(\frac{1}{\pi-\theta}\right)n^{-1/2}(\pi-\theta)^{-\beta-1/2}d\theta + \right.$$
$$\left. + \int_{\pi-c/n}^{\pi}(\pi-\theta)^{\nu+\beta-1/2}\varphi\left(\frac{1}{\pi-\theta}\right)n^{\beta}\,d\theta\right) \leqq \tag{4.4}$$
$$\leqq \begin{cases} K_5\left(n^{-1/2}+n^{-\nu-1/2}\varphi(n)\right) & \text{for } \nu>0 \text{ and } \dfrac{1}{2}>\beta\geqq-\dfrac{1}{2} \\ K_5 n^{-\nu-1/2}\varphi(n) & \text{for } 0>\nu>-\beta-\dfrac{1}{2} \text{ and } \dfrac{1}{2}>\beta>-\dfrac{1}{2} \end{cases}$$
$$\leqq K n^{-\delta+1/2}\varphi(n) \qquad \text{for } \nu\geqq\delta-1.$$

Since $\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)a_n$ converges, so does $\sum_{n=1}^{\infty} n^{-\delta+\alpha+1/2}\varphi(n)d_n$ by (2.1). Hence, if we put $q_n=d_n$ and $\mu=-\delta+\alpha+3/2$ in Lemma 3, then $\sum_{n=1}^{\infty} n^{-\delta+\alpha+3/2}\varphi(n)\Delta d_n$ converges. Now, by (4.2), (4.3) and (4.4), we have, for $\nu\neq 0$, $\nu\geqq\delta-1$, $\nu>-\beta-1/2$ and $1/2>\beta\geqq-1/2$,

$$\int_0^\pi |f(\theta)|\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\nu-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq$$
$$\leqq K + K_8\sum_{n=N}^{\infty} n^{-\delta+\alpha+3/2}\varphi(n)\Delta d_n < \infty .$$

Hence the case $\nu=0$ and $1/2>\beta>-1/2$ is clear. Thus (i) is proved.

Next, in order to prove (ii) it is enough to notice that for $\beta \geqq 1/2$

$$\int_{\pi/2}^{\pi} \theta^{\delta+\alpha-1/2}(\pi-\theta)^{\delta-\beta-5/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)|P_n^{(\alpha+1,\beta)}(\cos\theta)|\varrho^{(\alpha,\beta)}(\theta)\,d\theta \leqq$$

$$\leqq K\left\{\int_{\pi/2}^{\pi-c/n}(\pi-\theta)^{\delta-\beta-5/2}\varphi\left(\frac{1}{\pi-\theta}\right)n^{-1/2}(\pi-\theta)^{-\beta-1/2}(\pi-\theta)^{2\beta+1}\,d\theta+\right.$$

$$\left.+\int_{\pi-c/n}^{\pi}(\pi-\theta)^{\delta-\beta-5/2}\varphi\left(\frac{1}{\pi-\theta}\right)n^{\beta}(\pi-\theta)^{2\beta+1}\,d\theta\right\} \leqq K_1 n^{-\delta+1/2}\varphi(n)$$

by (S4), (2.6) and (S2). Thus Theorem 3 is proved.

5. Proofs of Theorem 4 and Corollary 1

For the proof of Theorem 4, we need the following three lemmas.

Lemma 4. *Let $\alpha, \beta \geqq -1/2$. Let $g(0) \in L([\theta, \pi]; \alpha, \beta)$ and let its Fourier—Jacobi series be*

$$g(\theta) \sim \sum_{n=0}^{\infty} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta).$$

Moreover, let

$$g_r(\theta) = \int_0^{\pi} g(v)\Psi_r(\theta, v)\varrho^{(\alpha,\beta)}(v)\,\mathrm{d}v = \sum_{n=0}^{\infty} b_n r^n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)$$

$$\text{for} \quad 0 \leqq \theta \leqq \pi \quad \text{and} \quad 0 \leqq r < 1,$$

where

$$\Psi_r(\theta, v) = \sum_{n=0}^{\infty} r^n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) P_n^{(\alpha,\beta)}(\cos v).$$

Then

$$\Psi_r(\theta, v) > 0 \quad \text{for} \quad 0 \leqq \theta,\ v \leqq \pi \quad \text{and} \quad 0 \leqq r < 1,$$

$$\int_0^{\pi} \Psi_r(\theta, v)\varrho^{(\alpha,\beta)}(v)\,dv = 1 \quad \text{for} \quad 0 \leqq \theta \leqq \pi \quad \text{and} \quad 0 \leqq r < 1,$$

and $g_r(\theta)$ converges to $g(\theta)$ as $r \to 1-0$ for almost every $0 \leqq \theta \leqq \pi$.

Lemma 4 is due to H. Bavinck [5; p. 4].

Lemma 5. *Let $\alpha, \beta \geqq -1/2$ and let $f(\theta) \in L([0, \pi]; \alpha, \beta)$. Suppose that $g(\theta)$ is a bounded and Lebesgue-measurable function on $[0, \pi]$. Then $f(\theta)g(\theta) \in L([0, \pi]; \alpha, \beta)$ and the formula*

$$\int_0^{\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,\mathrm{d}\theta = \lim_{r\to 1-0} \sum_{n=0}^{\infty} a_n b_n \omega_n^{(\alpha,\beta)} r^n$$

holds, where a_n, b_n denote the Fourier—Jacobi coefficients of the functions f, g respectively.

Proof. We have clearly $g(\theta)\in L([0,\pi];\alpha,\beta)$ and $f(\theta)g(\theta)\in L([0,\pi];\alpha,\beta)$. We define $g_r(\theta)$ and $\Psi_r(\theta, v)$ as in Lemma 4. Since $g(\theta)$ is bounded and Lebesgue-measurable, we have, by Lemma 4,

$$|g_r(\theta)| \leqq \int_0^\pi |g(v)|\Psi_r(\theta, v)\varrho^{(\alpha,\beta)}(v)\,dv \leqq \sup_{0\leqq v\leqq\pi} |g(v)| \quad \text{for} \quad 0\leqq\theta\leqq\pi \quad \text{and} \quad 0\leqq r<1,$$

and further $g_r(\theta)$ converges to $g(\theta)$ as $r\to 1-0$ for almost every $0\leqq\theta\leqq\pi$. Hence

$$\int_0^\pi f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \lim_{r\to 1-0}\int_0^\pi f(\theta)g_r(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \lim_{r\to 1-0}\sum_{n=0}^\infty a_n b_n \omega_n^{(\alpha,\beta)} r^n.$$

Thus Lemma 5 is proved.

Lemma 6. *If $H(\theta)$ is a non-negative and non-increasing function on* $(0,\infty)$ *and is Lebesgue-integrable in any finite interval, then*

$$\Big|\int_0^v H(\theta)\cos\theta\,d\theta\Big| \leqq \int_0^{\pi/2} H(\theta)\,d\theta \quad \text{and} \quad 0\leqq \int_0^v H(\theta)\sin\theta\,d\theta \leqq \int_0^\pi H(\theta)\,d\theta$$

for any $v>0$.

Lemma 6 is due to M. and S. Izumi [8; Lemma 1 and 2].

Proof of Theorem 4. By assumption, it is clear that $g(\theta)$ is bounded and Lebesgue-integrable on $[\eta', \eta'']$ for any $0<\eta'<\eta''<\pi$. Hence, since $f(\theta)\in L([0,\pi];$ $\alpha,\beta)$, we have, by Lemma 5,

$$\int_{\eta'}^{\eta''} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \lim_{r\to 1-0}\sum_{n=0}^\infty a_n\omega_n^{(\alpha,\beta)}r^n\int_{\eta'}^{\eta''} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta.$$

First we shall prove that $\int_{\eta'}^{\eta''} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$ converges to zero as η' and η'' tend to zero, i.e., that $\int_{\to 0}^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$ exists and is finite. Let $0<\eta'<\eta''\leqq$ $\leqq\pi/2$. We have, for any positive integer N,

$$\int_{\eta'}^{\eta''} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta =$$

$$= \lim_{r\to 1-0}\left\{\sum_{n=0}^N a_n\omega_n^{(\alpha,\beta)}r^n\int_{\eta'}^{\eta''} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta + \sum_{n=N+1}^\infty a_n\omega_n^{(\alpha,\beta)}r^n\big(I(\eta'')-I(\eta')\big)\right\}, \tag{5.1}$$

where

$$I(\eta) = \int_0^\eta g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \quad \text{for} \quad 0<\eta\leqq\pi/2.$$

We consider the case $\eta < c/n \leqq \pi/2$, where c is the positive constant in (2.6). Since $\theta^{\alpha+1/2}g(\theta)$ is non-negative, Lebesgue-integrable and non-increasing on $(0, \pi/2)$, we have, by (2.6),

$$|I(\eta)| \leqq K \int_0^{1/n} g(\theta) n^{\alpha} \left(\sin \frac{\theta}{2}\right)^{2\alpha+1} d\theta \leqq K_1 n^{-1/2} \int_0^{1/n} g(\theta)\theta^{\alpha+1/2}\, d\theta \quad \text{for} \quad \eta < \frac{c}{n} \leqq \frac{\pi}{2}. \tag{5.2}$$

Next we consider the case $c/n \leqq \eta$. We set $\lambda = (\alpha+\beta+1)/2$ and $\zeta = \alpha + 1/2$. By (2.8), we have

$$\sqrt{\omega_n^{(\alpha,\beta)}}\, I(\eta) = (D + D^* n^{-1}) T_1(\eta) + T_2(\eta) + O(n^{-2}) T_3(\eta), \tag{5.3}$$

where

$$T_1(\eta) = \int_{c/n}^{\eta} g(\theta) \left(\sin \frac{\theta}{2}\right)^{\alpha+1/2} \left(\cos \frac{\theta}{2}\right)^{\beta+1/2} \cos\left\{(n+\lambda)\theta - \frac{\pi\zeta}{2}\right\} d\theta,$$

$$T_2(\eta) = \int_{c/n}^{\eta} g(\theta) \left(\sin \frac{\theta}{2}\right)^{\alpha+1/2} \left(\cos \frac{\theta}{2}\right)^{\beta+1/2} \left(E n^{-1} \cot \frac{\theta}{2} + E^* n^{-1} \tan \frac{\theta}{2}\right) \times$$

$$\times \sin\left\{(n+\lambda)\theta - \frac{\pi\zeta}{2}\right\} d\theta,$$

$$T_3(\eta) = \int_{c/n}^{\eta} g(\theta) \left(\sin \frac{\theta}{2}\right)^{\alpha-3/2} \left(\cos \frac{\theta}{2}\right)^{\beta-3/2} d\theta.$$

Since $\theta^{\alpha+1/2}g(\theta)$ is non-negative and non-increasing on $(0, \pi/2)$, so is $(\sin \theta/2)^{\alpha+1/2} g(\theta)$. Now, without loss of generality, we may assume that $\cos \lambda\theta$ is positive and decreasing on $[0, \eta]$. Then, from Lemma 6,

$$\left| \int_{c/n}^{\eta} g(\theta) \left(\sin \frac{\theta}{2}\right)^{\alpha+1/2} \left(\cos \frac{\theta}{2}\right)^{\beta+1/2} \cos \lambda\theta \cos n\theta\, d\theta \right| \leqq$$

$$\leqq K_2 \int_0^{1/n} g(\theta) \left(\sin \frac{\theta}{2}\right)^{\alpha+1/2} \left(\cos \frac{\theta}{2}\right)^{\beta+1/2} \cos \lambda\theta\, d\theta \leqq K_3 \int_0^{1/n} \theta^{\alpha+1/2} g(\theta)\, d\theta.$$

Similary we have, by the second mean value theorem for integrals and Lemma 6,

$$|T_1(\eta)| \leqq K_4 \int_0^{1/n} \theta^{\alpha+1/2} g(\theta)\, d\theta \quad \text{for} \quad \frac{c}{n} \leqq \eta. \tag{5.4}$$

Similarly we get, for $cn^{-1} \leqq \eta$, with $\frac{c}{n} \leqq \psi$, $\psi' \leqq \eta$,

$$|T_2(\eta)| = \left|\left(\frac{E}{n}\cot\frac{c}{2n}\right)\int_{c/n}^{\psi} g(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\sin\left\{(n+\lambda)\theta-\frac{\pi\zeta}{2}\right\}d\theta+\right.$$

$$(5.5)\quad \left.+\left(\frac{E^*}{n}\tan\frac{\eta}{2}\right)\int_{\psi'}^{\eta} g(\theta)\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\sin\left\{(n+\lambda)\theta-\frac{\pi\zeta}{2}\right\}d\theta\right| \leqq$$

$$\leqq K_5\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta.$$

Hence, by (5.3), (5.4), (5.5) and (2.2), we obtain

$$|I(\eta)| \leqq K_6 n^{-1/2}\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta \quad \text{for} \quad \frac{c}{n} \leqq \eta. \tag{5.6}$$

Thus, by (5.2) and (5.6),

$$|I(\eta)| \leqq K_7 n^{-1/2}\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta \quad \text{for} \quad 0<\eta\leqq\frac{\pi}{2} \quad \text{and all} \quad n \geqq \left[\frac{2c}{\pi}\right]+1.$$

In this estimation, we can replace η by η' or η'', $0<\eta'<\eta''\leqq\pi/2$. Hence, by (5.1), (2.2) and (2.6), we have

$$\left|\int_{\eta'}^{\eta''} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta\right| \leqq$$

$$(5.7)\quad \leqq K_8\sum_{n=1}^{N}|a_n|(n+1)^{\alpha+1}\int_{\eta}^{\eta'} g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta + K_9\sum_{n=N+1}^{\infty}n^{1/2}|a_n|\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta.$$

We have $\sum_{n=1}^{\infty} n^{1/2}|a_n|\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta<\infty$ by assumption, so for any $\varepsilon>0$ we can take N so that the second term on the right-hand side is less than $\varepsilon/2$, and then take η' and η so near to zero that the first term on the right-hand side is less than $\varepsilon/2$. Then we have

$$\left|\int_{\eta'}^{\eta''} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta\right| < \varepsilon$$

for all η' and η'' sufficiently near zero. Thus,

$$\lim_{\eta\to+0}\int_{\eta}^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \int_{\to 0}^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \tag{5.8}$$

exists and is finite. Furthermore we have, by (5.1) and (5.7),

$$\int_{\to 0}^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta =$$

$$(5.9)\qquad \begin{aligned} &= \lim_{\eta\to+0}\lim_{r\to1-0}\sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)}r^n\int_{\eta}^{\pi/2} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \\ &= \lim_{\eta\to+0}\sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)}\int_{\eta}^{\pi/2} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{\alpha,\beta)}(\theta)\,d\theta = \\ &= \sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)}\int_{0}^{\pi/2} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta, \end{aligned}$$

where the last series converges absolutely. Similarly we can prove that

$$(5.10)\qquad \lim_{\eta\to\pi-0}\int_{\pi/2}^{\eta} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \int_{\pi/2}^{\to\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

exists and is finite, and that

$$(5.11)\qquad \int_{\pi/2}^{\to\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)}\int_{\pi/2}^{\pi} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta,$$

where the series converges absolutely. Hence, from (5.8), (5.9), (5.10) and (5.11), $f(\theta)g(\theta)\in R([0,\pi];\alpha,\beta)$ and Parseval's formula

$$\int_{\to 0}^{\to\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \sum_{n=0}^{\infty} a_n\omega_n^{(\alpha,\beta)}\int_{0}^{\pi} g(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \sum_{n=0}^{\infty} a_n b_n\omega_n^{(\alpha,\beta)}$$

holds, where the last series converges absolutely. Thus Theorem 4 is proved.

Proof of Corollary 1. We notice that the function $\theta(\pi-\theta)$ is increasing on $0\leqq\theta\leqq\pi/2$ and decreasing on $\pi/2\leqq\theta\leqq\pi$. If we put

$$g(\theta) = \theta^{\delta-\alpha-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\quad\text{for}\quad 0<\theta\leqq\frac{\pi}{2},$$

and

$$g(\theta) = (\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\quad\text{for}\quad \frac{\pi}{2}<\theta<\pi$$

in Theorem 4, then

$$\sum_{n=1}^{\infty} n^{1/2}|a_n|\Big(\int_0^{1/n}\theta^{\alpha+1/2}g(\theta)\,d\theta+\int_0^{1/n}\theta^{\beta+1/2}g(\pi-\theta)\,d\theta\Big)\leqq K\sum_{n=1}^{\infty} n^{-\delta+1/2}\varphi(n)|a_n|<\infty.$$

Hence we have $f(\theta)g(\theta)\in R([0,\pi];\alpha,\beta)$ by Theorem 4. Now, for any $\varepsilon>0$, there is a positive number $\Theta\leqq\pi/2$ such that

$$\left|\int_{\eta}^{\eta'} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta\right|<\varepsilon \quad \text{for all } \eta \text{ and } \eta',\quad 0<\eta<\eta'<\Theta.$$

Thus, by the second mean value theorem for integrals, we have, with $\eta\leqq\Psi\leqq\eta'$,

$$\left|\int_{\eta}^{\eta'} f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{\alpha,\beta}(\theta)\,d\theta\right|=$$

$$=(\pi-\eta')^{\delta-\beta-3/2}\left|\int_{\psi}^{\eta'} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta\right|\leqq\left(\frac{\pi}{2}\right)^{\delta-\beta-3/2}\varepsilon.$$

Hence

$$\int_{\to 0}^{\pi/2} f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

exists and is finite. Similarly,

$$\int_{\pi/2}^{\to\pi} f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta$$

exists and is finite. Thus,

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-3/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right)\in R([0,\pi];\alpha,\beta)$$

and Corollary 1 is proved.

6. Proofs of Theorem 5 and Corollaries 2, 3

For the proof of Theorem 5, we require the following two lemmas.

Lemma 7. *If* $\{q_n\}$ *is a non-negative and non-increasing sequence, then, for* $0\leqq m\leqq N\leqq\infty$, $0\leqq\theta\leqq\pi$ *and any real numbers* u, v, *we have*

$$\left|\sum_{n=m}^{N} q_n e^{i\{(n+u)\theta+v\}}\right|\leqq\sum_{n=0}^{[\theta^{-1}]} q_n \quad \text{for any } m, \text{ and } \quad \leqq K\theta^{-1}q_m \quad \text{for} \quad m\geqq\left[\frac{K_1}{\theta}\right].$$

Lemma 7 is due to L. MacFadden [10].

Lemma 8. Let $\alpha,\beta\geqq-1/2$. *Suppose that the Jacobi series* (1.5) *satisfies conditions* (J5), (J6) *and* (J7). *Define* $G(\theta)$ *as in* (1.6). *Then the series* (1.5) *converges* *to a function* $g(\theta)$ *in* $0<\theta<\pi$, *and the following four inequalities hold:*

(6.1) $$\left|\sum_{n=0}^{m} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqq K_1 G(\theta) \quad \text{for any } m \text{ and } \theta,\ 0 \leqq \theta \leqq \pi,$$

(6.2) $$\left|\sum_{n=m}^{N} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqq K_2 \theta^{-\alpha-3/2} m^{1/2} b_m \leqq K_3 g(\theta)$$

$$\text{for any} \quad m,\ N,\ \ N \geqq m \geqq \left[\frac{c}{\theta}\right]+1, \quad 0<\theta \leqq \frac{\pi}{2},$$

(6.3) $$\left|\sum_{n=m}^{N} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqq K_4 (\pi-\theta)^{-\beta-1/2} m^{1/2} b_m \leqq K_5 G(\theta)$$

$$\text{for any} \quad m,\ N,\ N \geqq m \geqq \left[\frac{c}{\pi-\theta}\right]+1, \quad \frac{\pi}{2}<\theta<\pi,$$

(6.4) $$|g(\theta)| \leqq G(\theta) \quad \text{for} \quad 0 \leqq \theta \leqq \pi,$$

where c *is a positive constant in* (2.6).

Proof. By (J5) and (J6), we notice $b_n \geqq 0$ for $n=0, 1, 2, \ldots$. Further, from (J6), we have

$$\sum_{n=1}^{\infty} |\Delta(n^{1/2} b_n)| < \infty \quad \text{and} \quad n^{1/2} b_n \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, by (J7) and Remark 2, the Jacobi series (1.5) satisfies (J1). Also, (1.5) satisfies clearly (J2). Thus, by Theorem 1, (1.5) converges to a function $g(\theta)$ in $0<\theta<\pi$.

First, we shall prove (6.2) and the case $0 \leqq \theta \leqq \pi/2$ of (6.1). We put $0<\theta \leqq \pi/2$. By (2.2), (2.6) and (J6), we have, for $0 \leqq m \leqq [c/\theta]$,

(6.5) $$\left|\sum_{n=0}^{m} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqq K \sum_{n=0}^{[c/\theta]} b_n (n+1)^{\alpha+1} \leqq K_1 G(\theta).$$

By Abel's transformation, we get, for $N \geqq m \geqq [c/\theta]+1$,

$$\sum_{n=m}^{N} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) = \sum_{n=m}^{N} (\Delta b_n^*) \frac{\alpha+1}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) P_n^{(\alpha+1,\beta)}(\cos\theta) -$$

(6.6) $$-b_m^* \cdot \frac{\alpha+1}{2m+\alpha+\beta} \omega_{m-1}^{(\alpha+1,\beta)} P_{m-1}^{(\alpha+1,\beta)}(1) P_{m-1}^{(\alpha+1,\beta)}(\cos\theta) +$$

$$+ b_{N+1}^* \cdot \frac{\alpha+1}{2N+\alpha+\beta+2} \omega_N^{(\alpha+1,\beta)} P_N^{(\alpha+1,\beta)}(1) P_N^{(\alpha+1,\beta)}(\cos\theta) = X_1 - X_2 + X_3,$$

say, where $b_n^* = b_n/P_n^{(\alpha,\beta)}(1)$. We have, for $N \geqq m \geqq \left[\frac{c}{\theta}\right]+1$,

$$(6.7)\qquad \begin{aligned} X_1 &= \sum_{n=m}^{N} \frac{\Delta(n^{1/2}b_n)}{n^{1/2}P_n^{(\alpha,\beta)}(1)} \cdot \frac{\alpha+1}{2n+\alpha+\beta+2}\,\omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1)\, P_n^{(\alpha+1,\beta)}(\cos\theta) + \\ &+ \sum_{n=m}^{N} (n+1)^{1/2} b_{n+1}\left(\Delta\,\frac{1}{n^{1/2}P_n^{(\alpha,\beta)}(1)}\right)\frac{\alpha+1}{2n+\alpha+\beta+2}\,\omega_n^{(\alpha+1,\beta)}\cdot P_n^{(\alpha+1,\beta)}(1) P_n^{(\alpha+1,\beta)}(\cos\theta) = \\ &= X_{1,1}+X_{1,2}, \end{aligned}$$

say. By (J6), (2.1), (2.2) and (2.6), we get

$$(6.8)\qquad |X_{1,1}| \leqq K_2\theta^{-\alpha-3/2}\sum_{n=m}^{N}\Delta(n^{1/2}b_n) \leqq K_2\theta^{-\alpha-3/2}m^{1/2}b_m.$$

By (1.1), (2.2), (2.7), (J6) and Lemma 7 (put $K_1=c$), we have

$$\begin{aligned} X_{1,2} &= \sum_{n=m}^{N}(n+1)^{1/2}b_{n+1}\left\{\frac{F}{n}+O\left(\frac{1}{n^2}\right)\right\}\left(\sin\frac{\theta}{2}\right)^{-\alpha-3/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}\times \\ &\times\left(A\cdot\cos\left\{\left(n+\frac{\alpha+\beta+2}{2}\right)\theta-\frac{\pi}{2}\left(\alpha+\frac{3}{2}\right)\right\}+(\sin\theta)^{-1}O(n^{-1})\right) = \\ &= O(1)\theta^{-\alpha-3/2}\sum_{n=m}^{N}(n+1)^{-1/2}b_{n+1}\cos\left\{\left(n+\frac{\alpha+\beta+2}{2}\right)\theta-\frac{\pi}{2}\left(\alpha+\frac{3}{2}\right)\right\}+ \end{aligned}$$

$$(6.9)\qquad \begin{aligned} &+O(1)\theta^{-\alpha-3/2}\sum_{n=m}^{N}(n+1)^{-3/2}b_{n+1}+O(1)\theta^{-\alpha-5/2}\sum_{n=m}^{N}(n+1)^{-3/2}b_{n+1}+ \\ &+O(1)\theta^{-\alpha-5/2}\sum_{n=m}^{N}(n+1)^{-5/2}b_{n+1} = \\ &= O(1)\theta^{-\alpha-3/2}\theta^{-1}(m+1)^{-1/2}b_{m+1}+O(1)\theta^{-\alpha-3/2}(m+1)^{-1/2}b_{m+1}+ \\ &+O(1)\theta^{-\alpha-5/2}(m+1)^{-1/2}b_{m+1}+O(1)\theta^{-\alpha-5/2}(m+1)^{-3/2}b_{m+1} = O(1)\theta^{-\alpha-3/2}m^{1/2}b_m \\ &\qquad \text{as}\quad m\to\infty, \end{aligned}$$

where F is a constant depending only on α and β. By (6.7), (6.8) and (6.9), we have

$$(6.10)\qquad |X_1| \leqq K_3\theta^{-\alpha-3/2}m^{1/2}b_m.$$

Next, from (6.6), (2.1), (2.2), (2.6) and (J6), we get

$$(6.11)\qquad |X_2| \leqq K_4 b_m^* m^{\alpha+1}m^{-1/2}\theta^{-\alpha-3/2} \leqq K_5\theta^{-\alpha-3/2}m^{1/2}b_m$$

and

$$(6.12)\qquad |X_3| \leqq K_6\theta^{-\alpha-3/2}N^{1/2}b_{N+1} \leqq K_7\theta^{-\alpha-3/2}m^{1/2}b_m.$$

Hence, from (6.6), (6.10), (6.11), (6.12) and (J6), we have, for $N \geqq m \geqq [c/\theta]+1$,

$$\left|\sum_{n=m}^{N} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right| \leqq K_8 \theta^{-\alpha-3/2} m^{1/2} b_m \leqq K_9 \theta^{-\alpha-2} b_{[c/\theta]+1} \leqq \tag{6.13}$$
$$\leqq K \sum_{n=0}^{[c/\theta]} (n+1)^{\alpha+1} b_n \leqq K_1 G(\theta).$$

Thus we get (6.2). By (6.5) and (6.13), we have the case $0<\theta\leqq\pi/2$ of (6.1). The case $\theta=0$ of (6.1) is trivial from (2.1) and (2.2). Hence we get the case $0\leqq\theta\leqq\pi/2$ of (6.1).

Secondly, we shall prove (6.3) and the case $\pi/2<\theta\leqq\pi$ of (6.1). We put $-1\leqq x<0$. By (1.2), (2.3) and Abel's transformation, we have, for $m=0, 1, 2, \ldots,$

$$\sum_{n=0}^{m} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x) = b_0 \omega_0^{(\beta,\alpha)} + \sum_{n=1}^{m} (-1)^n b_n \omega_n^{(\beta,\alpha)} P_n^{(\beta,\alpha)}(-x) =$$
$$= b_0 \omega_0^{(\beta,\alpha)} + \sum_{n=1}^{m} \left\{\sum_{s=1}^{n} (-1)^s\right\} \Delta \{b_n \omega_n^{(\beta,\alpha)} P_n^{(\beta,\alpha)}(-x)\} + \tag{6.14}$$
$$+ \left\{\sum_{s=1}^{m} (-1)^s\right\} b_{m+1} \omega_{m+1}^{(\beta,\alpha)} P_{m+1}^{(\beta,\alpha)}(-x) = b_0 \omega_0^{(\beta,\alpha)} + Y_1 + Y_2,$$

say. From (2.2) and (2.5), we get, for $n=1, 2, \ldots,$

$$\Delta\left(b_n \omega_n^{(\beta,\alpha)} P_n^{(\beta,\alpha)}(-x)\right) =$$
$$= 2\left(b_n n \left\{1+O\left(\frac{1}{n}\right)\right\} P_n^{(\beta,\alpha)}(-x) - b_{n+1}(n+1)\left\{1+O\left(\frac{1}{n}\right)\right\} P_{n+1}^{(\beta,\alpha)}(-x)\right) =$$
$$= 2\left(b_n\{(n+\beta+1) P_n^{(\beta,\alpha)}(-x) - (n+1) P_{n+1}^{(\beta,\alpha)}(-x)\} + O(1) b_n P_n^{(\beta,\alpha)}(-x) + \right. \tag{6.15}$$
$$\left. + O(1) b_{n+1} P_{n+1}^{(\beta,\alpha)}(-x) - (\beta+1) b_n P_n^{(\beta,\alpha)}(-x) + (\Delta b_n)(n+1) P_{n+1}^{(\beta,\alpha)}(-x)\right) =$$
$$= 2\left\{b_n \left(n+\frac{\alpha+\beta+2}{2}\right)(1+x) P_n^{(\beta+1,\alpha)}(-x) + O(1) b_n P_n^{(\beta,\alpha)}(-x) + \right.$$
$$\left. + O(1) b_{n+1} P_{n+1}^{(\beta,\alpha)}(-x) + (\Delta b_n)(n+1) P_{n+1}^{(\beta,\alpha)}(-x)\right\}.$$

By Abel's transformation, we have, for $m=1, 2, \ldots,$

$$\sum_{n=1}^{m} n^{\beta+1} \Delta b_n = \sum_{n-1}^{m-1} (-b_{n+1}) \Delta n^{\beta+1} + b_1 - b_{m+1} m^{\beta+1} \leqq K \sum_{n=1}^{m} n^\beta b_n. \tag{6.16}$$

By (J6), the sequence $\{b_n\}$ is non-increasing. From (6.14), (6.15), (2.1) and (6.16), we get, for $m=1, 2, \ldots,$

$$(6.17)\quad \begin{aligned} |Y_1| &\leqq K\left\{(1+x)\sum_{n=1}^{m} n^{\beta+2}b_n+\sum_{n=1}^{m} n^{\beta}b_n+\sum_{n=1}^{m} n^{\beta+1}\Delta b_n\right\} \leqq \\ &\leqq K_1\left\{(1+x)\sum_{n=0}^{m}(n+1)^{\beta+2}b_n+\sum_{n=0}^{m}(n+1)^{\beta}b_n\right\}. \end{aligned}$$

By (6.14), (2.1), (2.2) and (J6), we get, for $m=1, 2, \ldots,$

$$(6.18)\quad |Y_2| \leqq K(m+1)^{\beta+1}b_{m+1} \leqq K(m+1)^{\beta+1}b_m \leqq K_1\sum_{n=0}^{m}(n+1)^{\beta}b_n.$$

Thus, from (6.14), (6.17) and (6.18), we have, for $-1\leqq x<0$ and $m=0, 1, 2, \ldots,$

$$(6.19)\quad \left|\sum_{n=0}^{m} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(x)\right| \leqq K\left\{(1+x)\sum_{n=0}^{m}(n+1)^{\beta+2}b_n+\sum_{n=0}^{m}(n+1)^{\beta}b_n\right\}.$$

Now we put $-1<x<0$, i.e., $x=\cos\theta$, $\pi/2<\theta<\pi$. Then, by (6.19), we have, for $0\leqq m\leqq[c/(\pi-\theta)]$,

$$(6.20)\quad \begin{aligned} \left|\sum_{n=0}^{m} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\right| &\leqq K\left\{\frac{(\pi-\theta)^2}{2}\sum_{n=0}^{[c/(\pi-\theta)]}(n+1)^{\beta+2}b_n+\sum_{n=0}^{[c/(\pi-\theta)]}(n+1)^{\beta}b_n\right\} \leqq \\ &\leqq K_1\sum_{n=0}^{[c/(\pi-\theta)]}(n+1)^{\beta}b_n \leqq K_2G(\theta) \qquad \left(\frac{\pi}{2}<\theta<\pi\right). \end{aligned}$$

On the other hand, we can prove (6.3) by the same method of estimation as in (6.13). By (6.20) and (6.3), we have the case $\pi/2<\theta<\pi$ of (6.1). When we put $x=-1$ in (6.19), we have, for $m=0, 1, 2, \ldots,$

$$\left|\sum_{n=0}^{m} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(-1)\right| \leqq K\sum_{n=0}^{m}(n+1)^{\beta}b_n \leqq KG(\pi).$$

Hence we get the case $\theta=\pi$ of (6.1). Thus we have the case $\pi/2<\theta\leqq\pi$ of (6.1).

We have proved (6.1). The inequality (6.4) is trivial from (6.1). Thus Lemma 8 is proved.

Proof of Theorem 5. By the first part of Lemma 8, Jacobi series (1.5) converges to a function $g(\theta)$ in $0<\theta<\pi$. Since $f(\theta)G(\theta)\in L([0, \pi]; \alpha, \beta)$, so does $f(\theta)g(\theta)$ by (6.4). We have, for $m=0, 1, 2, \ldots,$

(6.21)

$$\sum_{n=0}^{m} a_nb_n\omega_n^{(\alpha,\beta)} = \int_0^{\pi} f(\theta)\left(\sum_{n=0}^{m} b_n\omega_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos\theta)\right)\varrho^{(\alpha,\beta)}(\theta)\,d\theta = \int_0^{\pi/2}+\int_{\pi/2}^{\pi} = W_1+W_2$$

say. Further we put

$$(6.22)\quad W_1 = \left(\int_0^{c/m} + \int_{c/m}^{\pi/2}\right) f(\theta)\left(\sum_{n=0}^{m} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right) \varrho^{(\alpha,\beta)}(\theta)\, d\theta = W_{1,1} + W_{2,2},$$

where c is a positive constant in (2.6). By (6.1) and assumption, we get

$$(6.23)\qquad |W_{1,1}| \leqq K \int_0^{c/m} |f(\theta)| G(\theta) \varrho^{(\alpha,\beta)}(\theta)\, d\theta \to 0 \quad \text{as} \quad m \to \infty.$$

We write

$$W_{1,2} = \int_{c/m}^{\pi/2} f(\theta) g(\theta) \varrho^{(\alpha,\beta)}(\theta)\, d\theta - \int_{c/m}^{\pi/2} f(\theta)\left(\sum_{n=m+1}^{\infty} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right) \varrho^{(\alpha,\beta)}(\theta)\, d\theta =$$

$$(6.24)\qquad = Z_1 - Z_2.$$

Since $f(\theta)g(\theta) \in L([0, \pi]; \alpha, \beta)$, we have

$$(6.25)\qquad Z_1 \to \int_0^{\pi/2} f(\theta) g(\theta) \varrho^{(\alpha,\beta)}(\theta)\, d\theta \quad \text{as} \quad m \to \infty.$$

Let $y_1 = 1/2$ and, for $n \geqq 2$,

$$y_n = n^{1/(2\alpha+3)} \quad \text{if} \quad b_n \leqq K n^{-3/2}, \quad \text{and} \quad y_n = \left(\frac{K}{n^{1/2} b_n}\right)^{1/(2\alpha+3)} \quad \text{if} \quad b_n > K n^{-3/2}.$$

Then

$$(6.26)\qquad n > y_n > 0, \quad y_n \to \infty \quad \text{and} \quad (n^{1/2} b_n) y_n^{\alpha+3/2} \to 0 \quad \text{as} \quad n \to \infty.$$

We put

$$(6.27)\quad Z_2 = \left(\int_{c/m}^{c/y_m} + \int_{c/y_m}^{\pi/2}\right) f(\theta)\left(\sum_{n=m+1}^{\infty} b_n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta)\right) \varrho^{(\alpha,\beta)}(\theta)\, d\theta = Z_{2,1} + Z_{2,2}.$$

Hence, by (6.27), (6.26) and (6.2) (let $N \to \infty$ in (6.2)),

$$(6.28)\qquad |Z_{2,1}| \leqq K \int_{c/m}^{c/y_m} |f(\theta)| G(\theta) \varrho^{(\alpha,\beta)}(\theta)\, d\theta \to 0 \quad \text{as} \quad m \to \infty.$$

Further, by (6.27), (6.2), (J6) and (6.26) (let $N \to \infty$ in (6.2)),

$$(6.29)\qquad \begin{aligned} |Z_{2,2}| &\leqq K \int_{c/y_m}^{\pi/2} |f(\theta)| \theta^{-\alpha-3/2} (m+1)^{1/2} b_{m+1} \varrho^{(\alpha,\beta)}(\theta)\, d\theta \leqq \\ &\leqq K_1 y_m^{\alpha+3/2} (m^{1/2} b_m) \int_{c/y_m}^{\pi/2} |f(\theta)| \varrho^{(\alpha,\beta)}(\theta)\, d\theta \to 0 \quad \text{as} \quad m \to \infty. \end{aligned}$$

By (6.27), (6.28) and (6.29), we have

$$(6.30)\qquad Z_2 \to 0 \quad \text{as} \quad m \to \infty.$$

From (6.24), (6.25) and (6.30), we get

$$W_{1,2} \to \int_0^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \quad \text{as} \quad m \to \infty. \tag{6.31}$$

Thus, by (6.31), (6.23) and (6.22),

$$W_1 \to \int_0^{\pi/2} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \quad \text{as} \quad m \to \infty. \tag{6.32}$$

When we consider W_2, we define y_n as follows:

(I) if $\beta \geqq 1/2$, let $y_1 = 1/2$ and, for $n \geqq 2$,

$$y_n = n^{1/(2\beta+1)} \quad \text{if} \quad b_n \leqq Kn^{-3/2}, \quad \text{and} \quad y_n = \left(\frac{K}{n^{1/2}b_n}\right)^{1/2(\beta+1)} \quad \text{if} \quad b_n > Kn^{-3/2},$$

(II) if $-1/2 \leqq \beta < 1/2$, let $y_1 = 1/2$ and, for $n \geqq 2$,

$$y_n = n^{1/2} \quad \text{if} \quad b_n \leqq Kn^{-3/2}, \quad \text{and} \quad y_n = \left(\frac{K}{n^{1/2}b_n}\right)^{1/2} \quad \text{if} \quad b_n > Kn^{-3/2}.$$

From (I) or (II), we have that $n > y_n > 0$, $y_n \to \infty$ and $(n^{1/2}b_n)y_n^{\beta+1/2} \to 0$ as $n \to \infty$. Now we shall obtain

$$W_2 \to \int_{\pi/2}^{\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta \quad \text{as} \quad m \to \infty$$

by the same method as in (6.32). Hence, combining this with (6.32) and (6.21),

$$\sum_{n=0}^{\infty} a_n b_n \omega_n^{(\alpha,\beta)} = \int_0^{\pi} f(\theta)g(\theta)\varrho^{(\alpha,\beta)}(\theta)\,d\theta.$$

Thus Theorem 5 is proved.

Proof of Corollary 2. Since $\beta \geqq \alpha > -1/2$ and $0 < \delta < \alpha + 1/2$, we have $-\delta + \alpha + 1/2 > 0$ and $-\delta + \beta - 1/2 > -1$. If we put $b_0 = 0$ and $b_n = n^{-\delta-1/2}\varphi(n)$ for $n = 1, 2, \ldots$, then $\{b_n\}$ satisfies (J5), (J6) and (J7). We get

$$\sum_{n=0}^{[1/\theta]} (n+1)^{\alpha+1} b_n \leqq K\theta^{\delta-\alpha-3/2}\varphi\left(\left[\frac{1}{\theta}\right]\right) \quad \text{for} \quad 0 \leqq \theta \leqq \frac{\pi}{2},$$

$$\sum_{n=0}^{[1/(\pi-\theta)]} (n+1)^{\beta} b_n \leqq K_1(\pi-\theta)^{\delta-\beta-1/2}\varphi\left(\left[\frac{1}{\pi-\theta}\right]\right) \quad \text{for} \quad \frac{\pi}{2} < \theta \leqq \pi.$$

Hence, from (1.8) and (S4),

$$G(\theta) \leqq K_2\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-1/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right).$$

Since

$$f(\theta)\theta^{\delta-\alpha-3/2}(\pi-\theta)^{\delta-\beta-1/2}\varphi\left(\frac{1}{\theta(\pi-\theta)}\right) \in L([0,\pi];\alpha,\beta),$$

so does $f(\theta)G(\theta)$. Thus, from Theorem 5, the series $\sum_{n=1}^{\infty} n^{-\delta+1/2}\,\varphi(n)\,a_n\omega_n^{(\alpha,\beta)}$ converges. From (2.2), we have

$$n=\frac{1}{2}\omega_n^{(\alpha,\beta)}\left\{1+\frac{B^*}{n}+O\left(\frac{1}{n^2}\right)\right\},$$

where B^* is a constant depending only on α and β. Hence $\sum_{n=1}^{\infty} n^{-\delta+1/2}\,\varphi(n)\,a_n$ converges. Thus Corollary 2 is proved.

Proof of Corollary 3. Since $\alpha\geqq\beta\geqq-1/2$ and $\alpha+1/2<\delta<\alpha+3/2$, we have $-1<-\delta+\alpha+1/2<0$ and $-\delta+\beta-1/2<1$. In Theorem 5 we put $b_0=0$ and $b_n=n^{-\delta-1/2}\varphi(n)$ for $n=1,2,\ldots$. Then it is sufficient to notice that

$$\sum_{n=0}^{[1/(\pi-\theta)]}(n+1)^{\beta}b_n\leqq K\qquad\text{for}\quad\frac{\pi}{2}<\theta\leqq\pi.$$

The rest of the proof is similar to that of Corollary 3.

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On the strong stability by Lyapunov's direct method

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Introduction

Consider the ordinary differential equation

$$\dot{x} = X(t, x), \tag{E}$$

where $t \in I = [0, \infty)$, and x belongs to the Euclidean n-space R^n. The function $X(t, x)$: $: \Gamma \to R^n$,

$$\Gamma = \{(t, x): t \in I, \|x\| < H\} \quad (0 < H = \text{const.}),$$

is continuous together with its first partial derivatives with respect to every component of x.

The unique solution of (E) through the point (t_0, x_0) denoted by $x(t) = x(t; t_0, x_0)$ is supposed to exist in I, provided that $\|x_0\|$ is sufficiently small. In addition, assume that $X(t, 0) \equiv 0$ for $t \in I$, i.e. $x = 0$ is a solution of (E), called the zero solution.

Recall first the following classical stability concepts. The zero solution of (E) is said to be

(i) *stable:* if given any $\varepsilon > 0$ and any $t_0 \in I$ there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|x_0\| < \delta(\varepsilon, t_0)$ implies $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \geqq t_0$.

(ii) *uniformly stable:* if given any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $t_0 \in I$, $\|x_0\| < \delta(\varepsilon)$ imply $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \geqq t_0$.

(iii) *asymptotically stable:* if it is stable and if given any $t_0 \in I$ there exists a $\sigma(t_0) > 0$ such that $\|x_0\| < \sigma(t_0)$ implies $\|x(t; t_0, x_0)\| \to 0$ as $t \to \infty$.

In this paper another type of stability, the so called strong stability will be investigated.

Definition 1. The zero solution of (E) is said to be *strongly stable* if given any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $t_0 \in I$, $\|x_0\| < \delta(\varepsilon)$ imply $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \in I$.

The concept of the strong stability was introduced by G. Ascoli [1] for linear systems. Def. 1 is taken from W. A. Coppel's monograph [2]. Obviously strong stability implies uniform stability, which, in turn, implies ordinary stability. Furthermore, the strong stability and the asymptotic stability are always incompatible.

At present some criteria for this type of stability are known for linear systems and for nonlinear systems of the form $\dot{x}=A(t)x+R(t, x)$, where $R(t, x)$ is small in a sense (see [2, p. 66]).

In general, the investigation of a given stability property by Lyapunov's direct method is based on principal theorems of the following type: the existence of a function $V(t, x)$: $\Gamma \to R$ with certain properties implies the desired stability property. In Sec. 1 we establish such a principal theorem for the strong stability, and we prove also the converse of this theorem. In Sec. 2 we give a sufficient condition for the strong stability by means of differential inequalities. This condition in several important cases may be easier to apply then the previous theorem. This can be seen in Sec. 3, where it is applied to the study of perturbed nonlinear differential equations and rheonomic mechanical systems under the action of potential forces

1. Lyapunov functions and the strong stability

According to the notations of W. HAHN's monograph [3], we shall say that *a function $a(r)$: $[0, H) \to R$ belongs to class K $(a(r) \in K)$* if it is continuous, strictly increasing on $[0, H)$ and $a(0)=0$.

Definition 1.1. A function $V(t, x)$: $\Gamma \to R$ having continuous first partial derivates in Γ, is said to be a *Lyapunov function* if $V(t, 0) \equiv 0$ for $t \in I$, and $V(t, x)$ is positive definite i.e. there exists a function $a(r) \in K$ such that $V(t, x) \geqq a(\|x\|)$ holds for $t \in I$ and for all x belonging to a certain ball $S_\lambda = \{x: \|x\| < \lambda\}$ $(\lambda > 0)$.

For every Lyapunov function $V(t, x)$ define the function

$$\dot{V}(t, x) = \sum_{i=1}^{n} \frac{\partial V(t, x)}{\partial x_i} X_i(t, x) + \frac{\partial V(t, x)}{\partial t}$$

which is said to be the *total derivative of $V(t, x)$ by virtue of equation (E)*. It is easy to see that for every solution $x(t)$ of (E)

$$\frac{d}{dt} V(t, x(t)) \equiv \dot{V}(t, x(t)) \qquad (t \in I). \tag{1.1}$$

Theorem 1.1. *The zero solution of (E) is strongly stable if and only if there exists a Lyapunov function $V(t, x)$ having the following properties:*

(1) *$V(t, x) \to 0$ as $x \to 0$ uniformly on I;*

(2) $\dot{V}(t, x) \equiv 0$ $((t, x) \in \Gamma)$.

Proof. *Sufficiency.* By the assumptions there are functions $a(r)$, $b(r)\in K$ such that

$$a(\|x\|) \leqq V(t, x) \leqq b(\|x\|) \tag{1.2}$$

for $t\in I$ and $\|x\|\leqq\lambda<H$, where λ is an appropriate positive constant. Given ε $(0<\varepsilon<\lambda)$, let $\delta(\varepsilon)>0$ be chosen so that $a(\varepsilon)>b(\delta(\varepsilon))$. Let $x(t; t_0, x_0)$ be a solution of (E) with $\|x_0\|<\delta(\varepsilon)$. Then, from (1.1), (1.2) and property (2), we get $a(\|x(t; t_0, x_0)\|)\leqq \leqq V(t, x(t; t_0, x_0))=V(t_0, x_0)\leqq b(\|x_0\|)\leqq b(\delta(\varepsilon))<a(\varepsilon)$. Hence $\|x(t; t_0, x_0)\|<\varepsilon$ for $t\in I$, i.e. the zero solution is strongly stable.

Necessity. Suppose that the zero solution is strongly stable. Then we shall prove that $V(t, x)=\|x(0; t, x)\|$ is a Lyapunov function with properties (1), (2).

The continuity of the partial derivatives of $V(t, x)$ follows from the smoothness of the right hand side of (E). Furthermore, to prove that the function $V(t, x)$ is positive definite it is sufficient to show that for every $\gamma>0$

$$m_\gamma = \inf\{V(t, x): t\in I, \|x\|\geqq\gamma\}>0.$$

Indeed, $m_\gamma\geqq\delta(\gamma)>0$ where $\delta(\gamma)$ corresponds to γ in the sense of Def. 1. Assuming the contrary, we have $(\bar{t}, \bar{x})\in\Gamma$ $(\|\bar{x}\|\geqq\gamma)$ such that $\|x(0; \bar{t}, \bar{x})\|=V(\bar{t}, \bar{x})<\delta(\gamma)$. Then, according to Def. 1, we have the estimation $\|x(t; \bar{t}, \bar{x})\|<\gamma$ for $t\in I$, which contradicts the inequality $\|x(\bar{t}; \bar{t}, \bar{x})\|=\|\bar{x}\|\geqq\gamma$. Consequently, $V(t, x)$ is a Lyapunov function.

Given $\varepsilon>0$ choose $\delta(\varepsilon)$ in the sense of Def. 1. If $\bar{t}\in I$ and $\|\bar{x}\|<\delta(\varepsilon)$, then $\|x(t; \bar{t}, \bar{x})\|<\varepsilon$ for $t\in I$. Consequently, the inequality $\|\bar{x}\|<\delta(\varepsilon)$ implies $\|x(0; \bar{t}, \bar{x})\|= =V(\bar{t}, \bar{x})<\varepsilon$ for $\bar{t}\in I$, which proves (1).

By (1.1) and the uniqueness of the solutions we have

$$\dot{V}(t, x) = \left[\frac{d}{d\tau} V(\tau, x(\tau; t, x))\right]_{\tau=t} = \left[\frac{d}{d\tau}\|x(0; t, x)\|\right]_{\tau=t} = 0$$

for all (t, x). Thus $V(t, x)$ has the property (2).

The theorem is proved.

This theorem is analogous to — but evidently independent of — K. P. PERSIDSKIĬ's well know theorem regarding the uniform stability (see [8]).

2. Differential inequalities and the strong stability

We begin by recalling two lemmas from the theory of differential inequalities (cf. [4]).

Lemma 2.1. *Suppose that the functions* $\omega_1(t, u):[t_0-T, t_0]\times\Omega\to R$, $\omega_2(t, v)$: $[t_0, t_0+T]\times\Omega\to R$ *are continuous, where* Ω *is an open interval in* R; t_0 *and* T *are positive constants. Let* $u^*(t)$, $v^*(t)$ *be the maximal solutions of the initial value problems*

$$\begin{cases} \dot{u} = \omega_1(t, u) \\ u(t_0) = \xi \end{cases} \qquad (t_0-T \leqq t < t_0;\ \xi\in\Omega),$$

$$\begin{cases} \dot{v} = \omega_2(t, v) \\ v(t_0) = \xi \end{cases} \qquad (t_0 < t \leqq t_0+T)$$

in $[t_0-T, t_0]$, $[t_0, t_0+T]$ *respectively.*

If the continuously differentiable function $w(t):[t_0-T, t_0+T]\to R$ *satisfies the inequalities* $w(t_0)\leqq\xi$;

$$\dot{w}(t) \geqq \omega_1(t, w(t)) \quad (t_0-T\leqq t \leqq t_0),$$

$$\dot{w}(t) \leqq \omega_2(t, w(t)) \quad (t_0 \leqq t \leqq t_0+T),$$

then $w(t)\leqq u^*(t)$ *for* $t\in[t_0-T, t_0]$ *and* $w(t)\leqq v^*(t)$ *for* $t\in[t_0, t_0+T]$.

Lemma 2.2. *Suppose that the function* $\omega(t, u_1, u_2):[t_0-T, t_0+T]\times\Omega_1\times\Omega_2\to R$ *is continuous and nondecreasing in* u_1, *where* Ω_1 *and* Ω_2 *are open intervals in* R; t_0 *and* T *are positive constants. Let* $u^*(t)$ *be the maximal solution of the initial value problem*

$$\begin{cases} \ddot{u} = \omega(t, u, \dot{u}) \quad (t_0-T\leqq t\leqq t_0+T), \\ u(t_0) = \xi,\ \dot{u}(t_0) = \eta \quad (\xi\in\Omega_1,\ \eta\in\Omega_2) \end{cases}$$

in $[t_0-T, t_0+T]$.

If the twice continuously differentiable function $w(t):[t_0-T, t_0+T]\to R$ *satisfies the conditions* $w(t_0)\leqq\xi$, $\dot{w}(t_0)=\eta$;

$$\ddot{w}(t) \leqq \omega(t, w(t), \dot{w}(t)) \quad (t_0-T\leqq t \leqq t_0+T),$$

then $w(t)\leqq u^*(t)$ *for* $t\in[t_0-T, t_0+T]$.

To formulate the main theorem of this section we have need of the following stability concept:

Definition. 2.1. The zero solution of (E) is said to be *uniformly stable at the right (at the left)* if for every $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that $t_0\in I$ and $\|x_0\|<\delta(\varepsilon)$ imply $\|x(t; t_0, x_0)\|<\varepsilon$ for all $t\geqq t_0$ (for all $t\in[0, t_0]$).

The concept of the uniform stability at the right corresponds to the concept of the uniform stability introduced by K. P. PERSIDSKIĬ ([see [8]). Here we give a necessary and sufficient condition for the uniform stability at the left.

Lemma 2.3. *The zero solution of* (*E*) *is uniformly stable at the left if and only if for every* $\varepsilon>0$ *the inequality*

$$\gamma(\varepsilon) = \inf \{\|x(t;\, t_0,\, x_0)\| : t_0\in I,\, t\geqq t_0,\, \|x_0\| = \varepsilon\} > 0 \tag{2.1}$$

holds.

Proof. *Necessity.* Suppose that the zero solution is uniformly stable at the left. We shall prove that for every $\varepsilon>0$ the inequality $\gamma(\varepsilon)\geqq\delta(\varepsilon)>0$ holds, where $\delta(\varepsilon)$ corresponds to ε in the sense of Def. 2.1. Assuming the contrary, we have $\varepsilon_0>0$ such that $\gamma(\varepsilon_0)<\delta(\varepsilon_0)$. Then, according to (2.1), there are $\bar t_0$, $\bar t$, $\bar x_0$ such that $\bar t_0\in I$, $\bar t\geqq\bar t_0$, $\|\bar x_0\|=\varepsilon_0$ and $\|x(\bar t;\, \bar t_0,\, \bar x_0)\|<\delta(\varepsilon_0)$. Hence, by virtue of Def. 2.1, we obtain $\varepsilon_0= =\|\bar x_0\|=\|x(\bar t_0;\, \bar t, x(\bar t; \bar t_0, \bar x_0))\|<\varepsilon_0$ which is a contradiction.

Sufficiency. Let $\gamma(\varepsilon)>0$ for every $\varepsilon>0$. We shall prove that the zero solution is uniformly stable at the left, namely given $\varepsilon>0$ the inequality $\|x_0\|<\gamma(\varepsilon)$ implies $\|x(t;\, t_0,\, x_0)\|<\varepsilon$ for all $t_0\in I$, $t\in[0,\, t_0]$.

Suppose that this is not true. Then there exist ε_0, $\bar t_0$, $\bar t$, $\bar x_0$ $(\varepsilon_0>0,\, 0\leqq\bar t<\bar t_0,$ $\|\bar x_0\|<\gamma(\varepsilon_0))$ such that $\|x(\bar t;\, \bar t_0,\, \bar x_0)\|=\varepsilon_0$. By (2.1) we get $\|\bar x_0\|=\|x(\bar t_0;\, \bar t,\, x(\bar t;\, \bar t_0,\, \bar x_0))\|\geqq$ $\geqq\gamma(\varepsilon_0)$ in contradiction with the assumption $\|\bar x_0\|<\gamma(\varepsilon_0)$.

The lemma is proved.

Remark 2.1. Lemma 2.3 shows that uniform stability at the left and asymptotic stability are always incompatible.

Comparing Def. 1 with Def. 2.1, we can easily obtain the following

Lemma 2.4. *The zero solution of* (*E*) *is strongly stable if and only if it is uniformly stable at the right and at the left, simultaneously.*

Remark 2.2. Let us now consider the linear system

$$\dot x = A(t)x, \tag{2.2}$$

where $A(t)$ is a square matrix whose elements are continuous functions for $t\in I$. Denote by $\Phi(t)$ the fundamental matrix of (2.2) with $\Phi(0)=E$, where E is the unit matrix. It is easy to see that in this case (2.1) becomes

$$\gamma(\varepsilon) = \inf\{\varepsilon\|\Phi(t_0)\Phi^{-1}(t)\|^{-1}:\; t_0\geqq 0,\; t\geqq t_0\} > 0.$$

Consequently, the zero solution of (2.2) is uniformly stable at the left if and only if the function $\|\Phi(t)\Phi^{-1}(s)\|$ is bounded on the set $0\leqq t\leqq s<\infty$.

Moreover, it is well known [2] that the zero solution of (2.2) is uniformly stable at the right if and only if the function $\|\Phi(t)\Phi^{-1}(s)\|$ is bounded on the set $0\leqq s\leqq t<\infty$.

Thus, the zero solution of (2.2) is strongly stable if and only if the functions $\|\Phi(t)\|$, $\|\Phi^{-1}(t)\|$ are bounded for $t\in I$ (cf. [2]), i.e. if and only if the zero solution of (2.2) is stable together with the zero solution of the adjoint system. (This latter property served originally as the definition of the strong stability for the linear system (see [1]).

Example 2.1. Let us consider the equation

$$\dot{u} = f(t)g(u) \quad (t\in I,\ u \geqq 0), \tag{2.3}$$

where the functions $f(t): I\to R$ and $g(u): I\to I$ are continuous; $g(0)=0$, $g(u)>0$ for $u>0$ and $\int\limits_0^\eta (g(u))^{-1}\,du=\infty$ $(0<\eta=\text{const.})$. Let $G(u;u_0):(0,\infty)\to(0,\infty)$ be the inverse of the function $\int\limits_{u_0}^{u}(g(s))^{-1}\,ds$ $(u_0>0)$. Then nontrivial solutions of (2.3) are given by the expression

$$u(t;\ t_0,u_0) = G\Big(\int\limits_{t_0}^{t} f(s)\,ds;\ u_0\Big). \tag{2.4}$$

Using (2.4), by Lemmas 2.3 and 2.4 it is easy to prove the following statement: *The zero solution of* (2.3) *is strongly stable if and only if*

$$\limsup_{t\to\infty}\Big|\int\limits_0^t f(s)\,ds\Big| < \infty.$$

Having these concepts and preliminary results, we state the following

Theorem 2.1. *Assume that there exists a Lyapunov function $V(t,x)$ satisfying the following conditions on Γ:*

(1) $V(t,x)\to 0$ *as* $x\to 0$ *uniformly on* I;

(2) $\omega_1(t, V(t,x))\leqq \dot{V}(t,x)\leqq \omega_2(t, V(t,x))$, *where the functions* $\omega_1(t,u)$, $\omega_2(t,v)$: $I\times I\to R$ *are continuous and* $\omega_1(t,0)\equiv\omega_2(t,0)\equiv 0$ for $t\in I$;

(3) *the zero solution* $u=0$ $(v=0)$ *of the equation* $\dot{u}=\omega_1(t,u)$ $(\dot{v}=\omega_2(t,v))$ *is uniformly stable at the left (at the right).*
Then the zero solution of (E) *is strongly stable.*

Proof. By the assumptions there exist functions $a(r)$, $b(r)\in K$ such that

$$a(\|x\|) \leqq V(t,x) \leqq b(\|x\|) \tag{2.5}$$

for all $t\in I$, $\|x\|<\lambda<H$, where $\lambda>0$ is an appropriate constant. Let ε $(0<\varepsilon<\lambda)$ be given. According to assumption (3), there exists a $\varkappa(\varepsilon)>0$ such that $0\leqq\eta<\varkappa(\varepsilon)$

implies

$$\begin{aligned} &0 \leqq u^*(t;\, t_0, \eta) < a(\varepsilon) \quad (t_0 \in I,\ 0 \leqq t \leqq t_0), \\ &0 \leqq v^*(t;\, t_0, \eta) < a(\varepsilon) \quad (t_0 \in I,\ t_0 \leqq t < \infty), \end{aligned} \tag{2.6}$$

where $u^*(t;\, t_0, \eta)$ $\big(v^*(t;\, t_0, \eta)\big)$ is the maximal solution of the equation $\dot{u} = \omega_1(t, u)$ $\big(\dot{v} = \omega_2(t, v)\big)$, passing through the point (t_0, η).

Let now $\delta(\varepsilon) > 0$ be chosen in such a way that $b\big(\delta(\varepsilon)\big) < \varkappa(\varepsilon)$. Further let $x(t)$ be a solution of (E) satisfying $\|x(t_0)\| < \delta(\varepsilon)$ for some $t_0 \in I$. Then, in view of (2.5), $V\big(t_0, x(t_0)\big) < \varkappa(\varepsilon)$. Applying Lemma 2.1, from (2.6) and assumption (2) we have

$$V\big(t, x(t)\big) \leqq u^*\big(t;\, t_0, V(t_0, x(t_0))\big) < a(\varepsilon) \quad (0 \leqq t \leqq t_0),$$

$$V\big(t, x(t)\big) \leqq v^*\big(t;\, t_0, V(t_0, x(t_0))\big) < a(\varepsilon) \quad (t_0 \leqq t < \infty),$$

i.e. $V\big(t, x(t)\big) < a(\varepsilon)$ for $t \in I$, from which, by (2.5) it follows that $\|x(t)\| < \varepsilon$ for $t \in I$. This means that the zero solution is strongly stable, q.e.d.

Suppose now that $X(t, x)$ has a continuous derivative with respect also to t, too. Then, analogously $\dot{V}(t, x)$, we define to $V(t, x)$ the function

$$\ddot{V}(t, x) = \sum_{i=1}^{n} \frac{\partial \dot{V}(t, x)}{\partial x_i} X_i(t, x) + \frac{\partial \dot{V}(t, x)}{\partial t},$$

provided that $V(t, x)$ has continuous second partial derivatives.

Theorem 2.2. *Assume that there exists a Lyapunov function $V(t, x)$ satisfying the following conditions on Γ:*

1) *$V(t, x) \to 0$ and $\dot{V}(t, x) \to 0$ as $x \to 0$ uniformly on I;*

2) *$\ddot{V}(t, x) \leqq \omega\big(t, V(t, x), \dot{V}(t, x)\big)$, where the function $\omega(t, u_1, u_2): I \times I \times R \to R$ is continuous and nondecreasing in u_1, and $\omega(t, 0, 0) \equiv 0$ for $t \in I$;*

3) *the zero solution of the equation $\ddot{u} = \omega(t, u, \dot{u})$ is strongly u-stable i.e. if given any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $t_0 \in I$, $0 \leqq u_0 < \delta(\varepsilon)$, $|\dot{u}_0| < \delta(\varepsilon)$ imply $0 \leqq u(t;\, t_0, u_0, \dot{u}_0) < \varepsilon$ for $t \in I$.*

Then the zero solution of (E) is strongly stable.

The proof of this theorem, based on Lemma 2.2, is similar to that of Theorem 2.1, and therefore it is omitted.

3. Applications

I. Let us consider the systems

$$\dot{x} = X(t, x) \tag{E}$$

$$\dot{y} = X(t, y) + R(t, y) \quad \big(R(t, 0) \equiv 0,\ t \in I\big). \tag{3.1}$$

Applying Theorem 2.1 we get conditions which guarantee that the strong stability of the zero solution of (E) is preserved under the perturbation $R(t, y)$.

We begin by recalling some notations and preliminary results. It is known [2] that the solution $x(t; t_0, x_0)$ of (E) is a differentiable function of t, t_0, x_0 on $I\times\Gamma$ and the $n\times n$ matrix

$$\Phi(t; t_0, x_0) = \left[\frac{\partial x_i(t; t_0, x_0)}{\partial x_{0j}}\right]_{i,j=1,2,\dots,n}$$

is the fundamental matrix with $\Phi(t_0; t_0, x_0)=E$ of the variational systems

$$\dot{z} = X_x\big(t; x(t; t_0, x_0)\big)z, \tag{3.2}$$

where the $n\times n$ matrix X_x is defined by

$$X_x(t, x) = \left[\frac{\partial X_i(t, x)}{\partial x_j}\right]_{i,j=1,2,\dots,n}.$$

For any real square matrix A, A^* denotes the transpose of A and $\lambda(A)$ denotes the smallest eigenvalue of the symmetric matrix $1/2(A+A^*)$. We use also the notation $\alpha(t)=\inf\{\lambda\big(X_x(t, x)\big): \|x\|<H\}$.

Applying WAŻEWSKI's inequality [5] to the system (3.2), we obtain

$$\|\Phi(t; t_0, x_0)\| \leqq L\exp\Big(\int_{t_0}^{t}\alpha(s)\,ds\Big) \quad (0\leqq t\leqq t_0;\ L=\text{const.}). \tag{3.3}$$

Theorem 3.1. *Let the solution $x=0$ of (E) be strongly stable. Then there are continuous functions $\gamma(t): I\to R$ $\big(\gamma(t)>0\big)$ and $g(r)\in K$ such that the inequality*

$$\|R(t, y)\| \leqq \gamma(t)g(\|y\|) \quad \big((t, y)\in\Gamma\big) \tag{3.4}$$

implies the strong stability of the solution $y=0$ of (3.1).

Proof. Let the solution $x=0$ of (E) be strongly stable. Let us consider the function $V(t, y)=\|x(0; t, y)\|^2$. In the proof of Theorem 1.1 it was verified that $[V(t, y)]^{1/2}$ is a Lyapunov function with the property $[V(t, y)]^{1/2}\to 0$ as $y\to 0$ uniformly on I. Consequently, there are functions $a(r)$, $b(r)\in K$ such that

$$a(\|y\|) \leqq V(t, y) \leqq b(\|y\|). \tag{3.5}$$

Choose continuous functions $\gamma(t)>0$ and $g(r)\in K$ such that

$$\int_0^\infty \gamma(t)\exp\Big[-\int_0^t\alpha(s)\,ds\Big]\,dt < \infty; \quad \int_0^\eta \frac{dr}{g\big(a^{-1}(r)\big)\sqrt{r}} = \infty \tag{3.6}$$

$(0<\eta=\text{const.})$, where by $a^{-1}(r)$ the inverse of the function $a(r)$ is denoted. We have to prove that assumption (3.4) with these functions $\gamma(t)$, $g(r)$ implies the strong stability of the solution $y=0$ of (3.1).

Using (z_1, z_2) to denote the scalar product of vectors $z_1, z_2 \in R^n$, for the total derivative $V'(t, y)$ of $V(t, y)$ by virtue of (3.1) we have

$$V'(t, y) = \dot{V}(t, y) + \sum_{j=1}^{n} \frac{\partial V(t, y)}{\partial y_j} R_j(t, y) =$$

(3.7)
$$= 2\sum_{j=1}^{n} \left(\frac{\partial}{\partial y_j} x(0;\, t, y),\, x(0;\, t, y) \right) R_j(t, y) =$$

$$= 2(\Phi(0;\, t, y) R(t, y),\, x(0;\, t, y)).$$

Applying the Cauchy inequality, from (3.3)—(3.5) and (3.7) we obtain the estimation

(3.8)
$$|V'(t, y)| \leqq 2L\gamma(t) \exp\left[-\int_0^t \alpha(s)\, ds\right] g(\|y\|)\, \|x(0;\, t, y)\| \leqq$$

$$\leqq 2L\gamma(t) \exp\left[-\int_0^t \alpha(s)\, ds\right] g(a^{-1}(V(t, y)))[V(t, y)]^{1/2}.$$

Now (3.5), (3.6), (3.8) and Example 2.1 show that we can apply Theorem 2.1 to equation (3.1). This concludes the proof.

It can be seen from (3.6) that the functions $\gamma(t)$, $g(r)$ depend, in general, on the unperturbed system (E). The following corollary shows that if (E) is linear then $\gamma(t)$, $g(r)$ are independent of (E).

Corollary 3.1. *Suppose that the functions $\gamma(t) > 0$ and $g(r) \in K$ have the properties*

(3.9)
$$\int_0^\infty \gamma(t)\, dt < \infty, \quad \int_0^\eta \frac{dr}{g(r)} = \infty \quad (0 < \eta = \text{const.}),$$

and let $R(t, y)$ satisfy assumption (3.4).

Then the strong stability of the solution $x=0$ of the system

(3.10)
$$\dot{x} = A(t)x$$

implies the strong stability of the solution $y=0$ of the perturbed system

(3.11)
$$\dot{y} = A(t)y + R(t, y).$$

Proof. In Remark 2.2 it was proved that the solution $x=0$ of (3.10) is strongly stable if and only if the fundamental matrix $\Phi(t)$ of (3.10) and its inverse are bounded.

in the matrix norm for $t \in I$. Therefore we may suppose that in (3.5) $a(r) \equiv c_1 r^2$ and $b(r) \equiv c_2 r^2$ with appropriate positive constants c_1, c_2. Furthermore, $\Phi(0; t, y) = \Phi^{-1}(t)$, consequently the estimation (3.8) has the form

$$|V'(t, y)| \leqq c_3 \gamma(t) g\left(\left[\frac{1}{c_1} V(t, y)\right]^{1/2}\right) [V(t, y)]^{1/2} \quad (c_3 = \text{const.}),$$

showing that in this case just the assumption (3.9) guarantees the applicability of Theorem 2.1.

The corollary is proved.

Corollary 3.1 contains one of W. A. COPPEL's theorems [2, Theorem 7, p. 67] as a special case.

II. Let

(3.12) $$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2, \ldots, n)$$

be the equations of a mechanical system in canonical form. Assume that $q=0$ is an equilibrium position.

Let us first suppose that (3.12) describes a conservative and scleronomic system. Then the Hamiltonian function $H(q, p)$ is the sum of the kinetic energy $T(q, p)$ and the potential energy $W(q)$; $T(q, p)$ is positive definite with respect to p, and $W(0)=0$. A well known theorem of J. L. LAGRANGE assures that the equilibrium position $q=0$ is stable if W has an isolated minimum there [7]. Theorem 1.1 shows that under the same condition the equilibrium position is not only stable but also strongly stable.

Consider now a rheonomic system under the action of potential forces, having Hamiltonian function of the form

(3.13) $$H(t, q, p) = \sum_{i, j=1}^{n} a_{ij}(t) p_i p_j + W(q) \quad (W(0) = 0),$$

where the scalar functions $a_{ij}(t)$ are continuously differentiable and bounded for $t \in I$.

Theorem 3.2. *If the Hamiltonian function* (3.13) *is positive definite and*

(3.14) $$\int_0^\infty \max\left\{\left|\frac{d}{dt} a_{ij}(t)\right| : i, j = 1, 2, \ldots, n\right\} \mathrm{d}t < \infty,$$

then the equilibrium position is strongly stable.

Proof. Since H is positive definite, there exist a number $a>0$ and a function $b(r)\in K$ such that $H(t,q,p)\geqq a\sum_{i=1}^{n} p_i^2+b(\|q\|)$. Furthermore,

$$\dot{H}=\dot{H}(t,q,p)=\frac{\partial H(t,q,p)}{\partial t}=\sum_{i,j=1}^{n}\frac{d}{dt}a_{ij}(t)p_ip_j;$$

hence we obtain the estimation

$$|\dot{H}|\leqq\frac{1}{2}\sum_{i,j=1}^{n}\left|\frac{d}{dt}a_{ij}(t)\right|(p_i^2+p_j^2)\leqq\frac{n}{a}\max\left\{\left|\frac{d}{dt}a_{ij}(t)\right|:\ i,j=1,2,\ldots,n\right\}H(t,q,p).$$

Moreover, the boundedness of the functions $a_{ij}(t)$ guarantees that $H(t,q,p)\to 0$ as $q\to 0$ and $p\to 0$ uniformly on I.

These properties of H, assumption (3.14), and Example 2.1 show that Theorem 2.1 can be applied to equations (3.12), and this concludes the proof.

*

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On the invariant subspace lattice $1+\omega^*$ (Corrigendum)

By DOMINGO A. HERRERO in Río Cuarto (Córdoba, Argentina)

The proofs of Theorems 1 and 2 of the author's paper [2] contain several errors, which come from implicitly assuming that if $\{b_n, \beta_n\}$ is a biorthogonal system with $\{\beta_n\}$ total, then the span of the b_n's is dense. (This is false, even for Hilbert spaces; see [3]). The author was unable to correct this point in Theorem 1; however, the proof given in that paper shows that the following weaker statement is actually true.

Theorem. *Let $\mathfrak{B}$ be a Banach algebra with identity and assume that* $\operatorname{Lat}\mathfrak{B}$ *(the lattice of closed left ideals of $\mathfrak{B}$) is the denumerable chain $\{\mathfrak{M}_n\}_{n=0}^{\infty}\cup\{(0)\}$, where $\mathfrak{M}_0=\mathfrak{B}$ and $\dim\mathfrak{M}_n/\mathfrak{M}_{n+1}=1$ for $n=0,1,2,\ldots$. Let t be an arbitrary element of $\mathfrak{M}_1\setminus\mathfrak{M}_2$; then t is quasinilpotent, $\mathfrak{B}$ is a (necessarily commutative) algebra of formal power series in t, the Gelfand spectrum of $\mathfrak{B}$ consists of a single point and $\mathfrak{M}_n=\operatorname{closure}(t^n\mathfrak{B})$, $n=0,1,2,\ldots$. Moreover, if $a=\sum_{n=0}^{\infty}c_nt^n\in\mathfrak{B}$, then there exist constants $\{C_n\}_{n=0}^{\infty}$ independent of a such that $|c_n|\leqq C_n\|a\|$.*

In other words, $\mathfrak{B}$ is the direct sum of a *generalized Banach algebra of power series* in the sense of S. GRABINER [1] and of $\mathbf{C}$ (the constant terms!). The author wishes to thank Professor SANDY GRABINER, who indicated the errors contained in [2] and also provided the correct statement of Theorem 1 given above.

In Corollary 3, it is necessary to make the following change: Instead of "$t\neq0$", we have to assume that "$t\neq\lambda e$ for all complex λ, where e denotes the identity of $\mathfrak{B}$". (In fact, if $\lambda_0\neq0$ is a root of the polynomial $p(z)$, then $p(\lambda_0e)=p(\lambda_0)e=0$, contradicting the thesis of the corollary.)

The result of Theorem 2 is correct, but the preliminary Lemma 5 needs several changes. Recall ([3, Chapter IX]) that $\{b_n, \beta_n\}_{n=0}^{\infty}$ (b_n belongs to the complex Banach space $\mathfrak{X}$ and β_n belongs to the dual space $\mathfrak{X}^*$) is a *biorthogonal system* if $\beta_n(b_m)=\delta_{nm}$ (Kronecker's delta); $\{b_n\}$ is a *(normalized) Markushevich basis* for $\mathfrak{X}$ if the b_n's are the first terms of a biorthogonal system such that $\{\beta_n\}$ is a total set of functionals in $\mathfrak{X}$ and $\{b_n\}$ span a dense linear manifold of $\mathfrak{X}$ (and $\|b_n\|=1$ for all n). For the existence and properties of Markushevich bases, the reader is referred to [3].

Replace Lemma 5 by the following

Lemma. *Let $\{b_n\}_{n=0}^{\infty}$ be a normalized Markushevich basis for the complex Banach space $\mathfrak{X}$ and let $\{\beta_n\}\subset\mathfrak{X}^*$ be chosen so that $\{b_n, \beta_n\}$ is a biorthogonal system. Let $0\leqq\varepsilon_n\leqq(2\|\beta_n\|)^{-1}$ and let $\{b_n', \beta_n'\}$ be a second biorthogonal system, with $\beta_n'=\beta_n+\varepsilon_{n+1}\beta_{n+1}$, $n=0, 1, 2, \ldots$. Then $\{b_n'\}$ is also a Markushevich basis for $\mathfrak{X}$.*

Proof. Clearly, $\|\beta_n\|\geqq\beta_n(b_n)=1$, so that $0\leqq\varepsilon_n\leqq 1/2$. That $\{\beta_n'\}$ is total in $\mathfrak{X}$ follows exactly as in [2, Lemma 5].

It only remains to show that the span of $\{b_n'\}$ is dense in $\mathscr{X}$. By induction over n, it is not difficult to see that

$$b_n'=b_n-\varepsilon_n b_{n-1}+\varepsilon_n\varepsilon_{n-1}b_{n-2}-\ldots+(-1)^n\varepsilon_n\varepsilon_{n-1}\ldots\varepsilon_2\varepsilon_1 b_0 \qquad (n=0, 1, 2, \ldots;\ b_{-1}=0),$$

so that b_n is a linear combination of $b_0', b_1', \ldots, b_n'$, whence the result follows.

By using this result it is very easy to obtain a correct proof of Lemma 6 and Theorem 2.

After the article [2] was published, a second proof of Theorem 2 was obtained by H. Radjavi and P. Rosenthal in [4], by using a different argument.

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Canonical number systems for complex integers

By I. KÁTAI and J. SZABÓ in Budapest

1. It is a well-known fact that every non-negative integer N has a unique representation of the form

$$N = a_0 + a_1 A + \ldots + a_k A^k, \tag{1.1}$$

where the integers a_j are selected from the set $\{0, 1, \ldots, A-1\}$, and A is an integer, $A \geqq 2$. Furthermore, choosing a negative integer $-A$ $(A \geqq 2)$, we can represent every integer N as a sum:

$$N = a_0 + a_1(-A) + \ldots + a_k(-A)^k, \quad 0 \leqq a_j \leqq A-1 \qquad (j = 0, 1, \ldots, k-1), \tag{1.2}$$

where a_j are integers. The representation (1.2) is also unique.

The number systems of negative base have some applications in the theory of computations.

The following question seems to be interesting: Given a Gaussian integer ϑ, can we represent every Gaussian integer α in the form

$$\alpha = r_0 + r_1 \vartheta + \ldots + r_k \vartheta^k \tag{1.3}$$

or not? Here $r_j \in \mathfrak{A}$, $\mathfrak{A}$ being a fixed complete residue system mod ϑ.

If the answer is affirmative, we say that $(\vartheta, \mathfrak{A})$ is a number system.

We shall investigate only the case $\mathfrak{A} = \mathfrak{A}_0$ where

$$\mathfrak{A}_0 = \{0, 1, \ldots, N(\vartheta) - 1\}, \tag{1.4}$$

and $N(\vartheta)$ denotes the "norm"

$$N(\vartheta) = \vartheta \cdot \bar{\vartheta} = (\operatorname{Re} \vartheta)^2 + (\operatorname{Im} \vartheta)^2.$$

It is known that for $\vartheta = -1 + i$, $(\vartheta, \mathfrak{A}_0)$ is a number system; see [1]

We prove:

Theorem 1. *$(\vartheta, \mathfrak{A}_0)$ is a number system if and only if*

a) $\operatorname{Re} \vartheta < 0$ *and* b) $\operatorname{Im} \vartheta = \pm 1$.

For $\vartheta = -A \pm i$ the representation of α in the form (1.3) *is unique.*

Theorem 2. *Let $\vartheta=-A\pm i$, z an arbitrary complex number. Then*

$$z = a_l\vartheta^l+\dots+a_0+\frac{a_{-1}}{\vartheta}+\frac{a_{-2}}{\vartheta^2}+\dots, \tag{1.5}$$

where $a_j\in\mathfrak{A}_0$ $(j=l, l-1, \dots, 0, -1, -2, \dots)$.

We do not assert the uniqueness of the representation of z in the form (1.5).

2. Proof of Theorem 1. *Necessity.* Let $\vartheta=A+Bi$. Then

$$\mathfrak{A}_0 = \{0, 1, \dots, A^2+B^2-1\}.$$

It is obvious that $\mathfrak{A}_0$ must be a complete residue system mod ϑ if $(\vartheta, \mathfrak{A}_0)$ is a number system. In the opposite case there is an α which is incongruent to k for every k in $\mathfrak{A}_0$, but from (1.3) $\alpha\equiv r_0 (\operatorname{mod} \vartheta)$, $r_0\in\mathfrak{A}_0$ follows, and this is a contradiction.

Suppose that $A>0$. We prove that $\alpha=(1-A)+iB=1-\bar{\vartheta}$ has no representation of type (1.3). Suppose in the contrary that

$$\alpha = r_0+r_1\vartheta+\dots+r_k\vartheta^k. \tag{2.1}$$

Let

$$\varrho = \alpha(1-\vartheta) = (1-A)^2+B^2 = A^2+B^2-2A+1.$$

Since $A\geqq 1$, we have $\varrho\in\mathfrak{A}_0$. From (2.1) we get

$$\varrho = r_0+(r_1-r_0)\vartheta+\dots+(r_k-r_{k-1})\vartheta^k-r_k\vartheta^{k+1}.$$

Hence $\varrho\equiv r_0 \bmod \vartheta$, and by $\varrho\in\mathfrak{A}_0$, $r_0\in\mathfrak{A}_0$ we get: $\varrho=r_0$. So

$$(r_1-r_0)\vartheta+\dots+(r_k-r_{k-1})\vartheta^k-r_k\vartheta^{k+1} = 0.$$

Hence it follows immediately that

$$r_1-r_0 = 0, \dots, r_k-r_{k-1} = 0, \quad r_k = 0,$$

whence $r_k=r_{k-1}=\dots=r_1=r_0=0$. Therefore $\varrho=0$, and so $A=1$, $B=0$. But it is obvious that $\vartheta=1$ is not a base of a number system. Similarly, $\vartheta=\pm i$ $(A=0, B=\pm 1)$ is not a base of a number system, either.

Let now $\operatorname{Im}\vartheta=B\neq\pm 1$. Let us take into account that B is a divisor of $\operatorname{Im}\vartheta^\nu$ $(\nu=1, 2, \dots)$. Hence, for an α of (1.3) we get:

$$\operatorname{Im}\alpha = r_1\operatorname{Im}\vartheta+\dots+r_k\operatorname{Im}\vartheta^k,$$

and so $B|\operatorname{Im}\alpha$. Consequently, (1.3) will not hold for $\alpha=i$ $(B\neq\pm 1)$.

Sufficiency. Let now $\vartheta=-A+i$ $(A\geqq 1)$. Then $\mathfrak{A}_0$ is a complete residue system mod ϑ as it is well known. Let us take into account, that

$$\vartheta^2+2A\vartheta+A^2+1 = 0. \tag{2.2}$$

Let $\alpha=E+Fi$ be an arbitrary Gaussian integer. Taking $D=F$, $C=E+AF$, we get

(2.3) $$\alpha = C+D\vartheta.$$

First we prove that every α has the form

(2.4) $$\alpha = U+V\vartheta+X\vartheta^2+Y\vartheta^3,$$

where U, V, X, Y are non-negative integers. From (2.2) we have

$$-1 = \vartheta^2+2A\vartheta+A^2.$$

Assuming that $C<0$ we can substitute C in (2.3) by

$$|C|\cdot\vartheta^2+2A|C|\cdot\vartheta+A^2|C|.$$

In the case $D<0$ we take a similar substitution, and get (2.4).

We shall use the following relation:

(2.5) $$A^2+1 = \vartheta^3+(2A-1)\vartheta^2+(A-1)^2\vartheta.$$

Let

(2.6) $$\alpha = d_0+d_1\vartheta+\ldots+d_k\vartheta^k \quad (k \geqq 3), \quad d_j \geqq 0 \quad (j = 0, \ldots, k).$$

Let

(2.7) $$t(\alpha, d) = d_0+d_1+\ldots+d_k;$$

$t(\alpha, d)$ is a non-negative integer, $t(\alpha, d)=0$ only if $\alpha=0$.

We take

$$d_0 = r_0+tN(\vartheta) = r_0+t(A^2+1),$$

$t\geqq 0$, integer, $0\leqq r_0\leqq A^2$. From (2.5) we have

(2.8) $$d_0 = r_0+t(A^2+1) = r_0+t(A-1)^2\vartheta+t(2A-1)\vartheta^2+t\vartheta^3.$$

We take the right hand side of (2.8) into (2.6). Then

$$\alpha = r_0+\big(d_1+t(A-1)^2\big)\vartheta+\big(d_2+t(2A-1)\big)\vartheta^2+(d_3+t)\vartheta^3+d_4\vartheta^4+\ldots+d_k\vartheta^k =$$

(2.9) $$= d_0^*+d_1^*\vartheta+\ldots+d_k^*\vartheta^k.$$

Since

$$-t(A+1)^2+t(A-1)^2+t(2A-1)+t = 0,$$

therefore

$$t(\alpha, d^*) = d_0^*+\ldots+d_k^* = t(\alpha, d), \quad d_j^* \geqq 0 \qquad (j = 0, \ldots, k).$$

Let

(2.10) $$\alpha_1 = d_1^*+d_2^*\vartheta+\ldots+d_k^*\vartheta^{k-1}.$$

We have

$$\alpha = \alpha_1 \vartheta + r_0 \quad (r_0 \in \mathfrak{A}_0), \tag{2.11}$$

$$t(\alpha_1, d^*) = d_1^* + d_2^* + \ldots + d_k^*.$$

It is obvious that $t(\alpha_1, d^*) < t(\alpha, d)$, when $r_0 \neq 0$. For $r_0 = 0$, $t(\alpha_1, d^*) = t(\alpha, d)$.

Now we write $t(\alpha, d) = t(\alpha)$, $t(\alpha_1, d^*) = t(\alpha_1), \ldots$. We repeat the algorithm (2.9), (2.11):

$$\alpha = \alpha_1 \vartheta + r_0, \quad \alpha_1 = \alpha_2 \vartheta + r_1, \quad \ldots, \quad \alpha_{j-1} = \alpha_j \vartheta + r_{j-1} \quad (r_i \in \mathfrak{A}_0).$$

Then $t(\alpha) \geqq t(\alpha_1) \geqq \ldots$ and $t(\alpha_i) > t(\alpha_{i+1})$ when $r_i \neq 0$. This process is terminated at the jth step if $\alpha_j = 0$. In this case we get

$$\alpha = r_0 + r_1 \vartheta + \ldots + r_{j-1} \vartheta^{j-1} \quad (r_i \in \mathfrak{A}_0).$$

Suppose that the process is not terminated. Then for a suitably large i

$$t(\alpha_i) = t(\alpha_{i+1}) = \ldots (\neq 0).$$

Hence

$$\alpha_i = \alpha_{i+1} \vartheta, \ldots \alpha_{i+k-1} = \alpha_{i+k} \vartheta$$

and, therefore, $\vartheta^k | \alpha_i$ $(k = 1, 2, \ldots)$. This holds only if $\alpha_i = 0$.

We proved the existence of the representation of α in the form (1.3).

Let us suppose now that there is an α wich has two different representations:

$$\alpha = r_0 + r_1 \vartheta + \ldots + r_k \vartheta^k = s_0 + s_1 \vartheta + \ldots + s_k \vartheta^k, \quad r_i, s_i \in \mathfrak{A}_0.$$

Then $0 = (r_0 - s_0) + (r_1 - s_1)\vartheta + \ldots + (r_k - s_k)\vartheta^k$ and therefore $r_0 \equiv s_0 \bmod \vartheta$; as r_0, $s_0 \in \mathfrak{A}_0$ we get $r_0 = s_0$. Dividing by ϑ, we get

$$0 = (r_1 - s_1) + \ldots + (r_k - s_k)\vartheta^{k-1}.$$

We repeat the argument. Finally we get:

$$r_0 = s_0, r_1 = s_1, \ldots, r_k = s_k.$$

We have proved the theorem for $\vartheta = -A + i$.

Let now $\vartheta = -A - i$. Using the theorem for $\bar{\vartheta} = -A + i$, we get

$$\bar{\alpha} = r_0 + r_1 \bar{\vartheta} + \ldots + r_k \bar{\vartheta}^k \quad (r_i \in \mathfrak{A}_0)$$

for every Gaussian integer $\bar{\alpha}$. Hence

$$\alpha = r_0 + r_1 \vartheta + \ldots + r_k \vartheta^k,$$

and so the theorem holds for $\vartheta = -A - i$, too.

3. Proof of Theorem 2. Let z be an arbitrary complex number, $z=x+iy$. Let

$$\vartheta^k = U_k+iV_k. \tag{3.1}$$

We have

$$z = \frac{z\vartheta^k}{\vartheta^k} = \frac{(x+iy)(U_k+iV_k)}{\vartheta^k} = \frac{C_k+D_k i}{\vartheta^k}+\frac{u_k+v_k i}{\vartheta^k}, \tag{3.2}$$

where C_k, D_k are rational integers, $|u_k|<1$, $|v_k|<1$. Let

$$z_k = \frac{C_k+iD_k}{\vartheta^k}, \quad \delta_k = \frac{u_k+iv_k}{\vartheta^k}. \tag{3.3}$$

It is obvious that $\delta_k \to 0$ $(k\to\infty)$, and so $z_k\to z$. Since C_k+iD_k is a Gaussian integer, by Theorem 1 we have

$$C_k+iD_k = a_t^*\vartheta^t+\ldots+a_0^*, \quad t = t(k). \tag{3.4}$$

First we prove that the sequence $t(k)-k$ $(k=1, 2, \ldots)$ has an upper bound. Indeed, from (3.4)

$$z_k = a_t^*\vartheta^{t-k}+\ldots+a_0^*\vartheta^{-k}.$$

Hence

$$a_t^*\vartheta^{t-k}+\ldots+a_k^* = z_k-\frac{a_{k-1}^*}{\vartheta}-\ldots-\frac{a_0^*}{\vartheta^k}, \tag{3.5}$$

and so

$$\begin{gathered}|a_t^*\vartheta^{t-k}+\ldots+a_k^*| \leqq |z_k|+\frac{a_{k-1}^*}{|\vartheta|}+\ldots+\frac{a_0^*}{|\vartheta|^k} \leqq \\ |z|+|\delta_k|+A^2\left(\frac{1}{|\vartheta|}+\frac{1}{|\vartheta|^2}+\ldots\right) \leqq |z|+|\delta_k|+\frac{A^2}{|\vartheta|-1}.\end{gathered} \tag{3.6}$$

Hence it follows that

$$|a_t^*\vartheta^{t-k}+\ldots+a_k^*| \leqq c, \tag{3.7}$$

$c=c(z)$ being a suitable positive constant.

Since the representation of Gaussian integers in the form (1.3) is unique, and the circle $|w|\leqq c$ contains only a finite set of Gaussian integers, therefore $t(k)-k$ has an upper bound. Let K be an integer, $t-k\leqq K$. Then we can write z_k as

$$z_k = a_K^{(k)}\vartheta^K+\ldots+a_0^{(k)}+\frac{a_{-1}^{(k)}}{\vartheta}+\frac{a_{-2}^{(k)}}{\vartheta^2}+\ldots, \tag{3.8}$$

where $a_j^{(k)}\in\mathfrak{A}_0$ $(j=K, K-1, \ldots, 0, -1, \ldots)$. Let $b_K\in\mathfrak{A}_0$ be an integer so that $a_K^{(k)}=$ $=b_K$ for infinitely many k. Let S_K be the subset of those integers k satisfying $a_K^{(k)}=$

$=b_k$. Suppose that $S_K, \dots, S_{l+1}$ is constructed $(S_k \supseteq \dots \supseteq S_{l+1})$. Then there is a $b_l \in \mathfrak{A}_0$, such that for infinitely many k in S_{l+1} $a_l^{(k)} = b_l$. Let S_l be the set of these k's. S_l has infinitely many elements. We repeat this argument for $K, K-1, \dots 0, -1, \dots$. Let

$$w = b_K \vartheta^K + \dots + b_0 + \frac{b_{-1}}{\vartheta} + \dots .$$

Let $k_1 < k_2 < \dots$ be an infinite sequence, so that

$$k_\nu \in S_{K-\nu+1} \quad (\nu = 1, 2, \dots).$$

Since

$$z_k = b_K \vartheta^K + \dots + b_{K-\nu+1} \vartheta^{K-\nu+1} + a_{K-\nu}^{(k_\nu)} \vartheta^{K-\nu} + \dots,$$

therefore

$$\lim_{\nu \to \infty} z_{k_\nu} = w.$$

Taking into account that $\lim_{k \to \infty} z_k = z$, we have $w = z$. Hence it follows that (3.9) is a suitable representation of z.

We have proved Theorem 2.

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Note on integral inequalities

By L. LEINDLER in Szeged

In [1] we proved the integral inequality

$$\int_{-\infty}^{\infty} \sup_{x/p+y/q=t} f(x)g(y)\,dt \geqq \Big(\int_{-\infty}^{\infty} f^p(x)\,dx\Big)^{1/p}\Big(\int_{-\infty}^{\infty} g^q(x)\,dx\Big)^{1/q} \tag{1}$$

for arbitrary non-negative measurable functions $f(x)$, $g(x)$ and for fixed p and q satisfying the conditions $1 \leqq p,\ q \leqq \infty$ and $1/p+1/q=1$, assuming that the left-hand side of (1) has sense.

Setting $F(x, y)=f(x)g(y)$ (1) can be written in the form

$$\int_{-\infty}^{\infty} \sup_{x/p+y/q=t} F(x, y)\,dt \geqq \Big(\int_{-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty} F^p(x, y)\,dx\Big\}^{q/p} dy\Big)^{1/q}. \tag{2}$$

Professor B. SZ.-NAGY raised the problem whether inequality (2) holds for an arbitrary non-negative measurable function $F(x, y)$ of two variables. The answer to this question is negative. A counter-example is yielded, say in the case $p=q=2$, by the function

$$F_1(x, y) = \begin{cases} 1 & \text{if} \quad 0 \leqq x \leqq 3 \quad \text{and} \quad -x+2 \leqq y \leqq -x+3, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, straightforward computation yields, in this case, the value 1/2 for the left-hand side, and the value $\sqrt{5/2}$ for the right-hand side, of (2).

However, instead of (2) we can prove the inequality

$$\max_{\alpha} \int_{-\infty}^{\infty} \sup_{\alpha x+(1-\alpha)y=t} F(x, y)\,dt \geqq \Big(\int_{-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty} F^p(x, y)\,dx\Big\}^{q/p} dy\Big)^{1/q},$$

where α runs over the two-point set $\{0, 1\}$.

More generally, we have the following

Theorem. *Let $f(x_1, x_2, \dots, x_m)$ be a non-negative measurable function and set*

$$J = \max J_i$$

where

$$J_i=\int_{-\infty}^{\infty} S_i(x_i)\,dx_i,\quad S_i(x_i)=\sup_{x_1,x_2,\dots,x_{i-1},x_{i+1},\dots,x_m} f(x_1,x_2,\dots,x_m)$$

$$(i=1,2,\dots,m).$$

Then we have

$$\left(\int_{-\infty}^{\infty}\left(\dots\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f^{p_1}(x_1,\dots,x_m)\,dx_1\right)^{p_2/p_1}dx_2\right)^{p_3/p_2}dx_3\dots\right)^{p_m/p_{m-1}}dx_m\right)^{1/p_m}\leqq J \tag{3}$$

for arbitrary numbers $p_1,p_2,\dots,p_m(\geqq 1)$ *with* $\sum_{i=1}^{m}1/p_i=1$.

Proof. It is clear that

$$J=\prod_{i=1}^{m}J^{1/p_i}\geqq\prod_{i=1}^{m}J_i^{1/p_i}=\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}S_1(x_1)\,dx_1\right)^{p_2/p_1}S_2(x_2)\,dx_2\right)^{1/p_2}\prod_{i=3}^{m}J_i^{1/p_i}=$$

$$=\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}S_1(x_1)\,S_2^{p_1/p_2}(x_2)\,dx_1\right)^{p_2/p_1}dx_2\right)^{1/p_2}\prod_{i=3}^{m}J_i^{1/p_i}=$$

$$=\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}S_1(x_1)\,S_2^{p_1/p_2}(x_2)\,dx_1\right)^{p_2/p_1}dx_2\right)^{p_3/p_2}S_3(x)\,dx_3\right)^{1/p_3}\prod_{i=4}^{m}J_i^{1/p_i}=$$

$$=\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}S_1(x_1)\,S_2^{p_1/p_2}(x_2)\,S_3^{p_1/p_3}(x_3)\,dx_1\right)^{p_2/p_1}dx_2\right)^{p_3/p_2}dx_3\right)^{1/p_3}\prod_{i=4}^{m}J_i^{1/p_i}.$$

It is easy to see that repeating this procedure we arrive at the inequality

$$J\geqq\left(\int_{-\infty}^{\infty}\left(\dots\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}S_1(x_1)\,S_2^{p_1/p_2}(x_2)\,S_3^{p_1/p_3}(x_3)\cdot\dots\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\dots\cdot S_m^{p_1/p_m}(x_m)\,dx_1\right)^{p_2/p_1}dx_2\right)^{p_3/p_2}\dots\right)^{p_m/p_{m-1}}dx_m\right)^{1/p_m}.$$

Since

$$1+p_1/p_2+\dots+p_1/p_m=p_1$$

and

$$S_i(x_i)\geqq f(x_1,x_2,\dots,x_m),$$

we thus get (3). This completes the proof.

Note that our theorem can be generalized from *the space* R to the space R^n. To prove this we have only to write $\mathbf{x}_i\in R^n$ instead of $x_i\in R$ throughout the proof.

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Tensor product of Hilbert lattices and free orthodistributive product of orthomodular lattices

By TAMÁS MATOLCSI in Szeged

I. Introduction

In probability theory one defines the product probability space of a set of probability spaces as the product measure space known from measure theory ([1], [2]). However, the meaning of the product probability space is not generally clarified in probability theory except for a particular case. The distribution space of a random variable is a probability space built on the real line in a natural way. Given a set of independent random variables, the product of their distribution spaces can be interpreted as the distribution space of the random variables together.

Physics uses probability theory in the description of physical phenomena. However, it does from a point of view which is somewhat different from that of the classical probability theory. In the simplest physical theory, in mechanics, one assigns to each physical system a so-called logic which is the analogue of the algebraic structure of events and one is concerned with a set of probability measures on the logic, called the states of the physical system ([3]). Independence of events has generally no sense in this case, because independence is formulated with respect to a *given* probability measure. Events or random variables independent for one probability measure can be not independent for another. States of a physical system change one into another and it would be too restrictive to define independence with respect to all sates.

Therefore we see that the definition of product probability spaces as stated in classical probability theory may not work in physics. Moreover, there is another difficulty: one cannot assume in general that the logic is a σ-field of subsets of a set ([3]).

Nevertheless, there is a need for something like product probability space. Namely, if we are given two physical systems, how can we get a new physical system consisting of these two together?

In this paper mathematical aspects will be studied and physics appears again only in Discussion.

In the following sections there are given a definition and a solution of a problem without any relevance to general probability theory. A probabilistic point of view is given only in Discussion, together with a general formulation of the problem.

II. Hilbert lattices

In the sequel a Hilbert space means a non-zero finite or countably infinite dimensional complex or real Hilbert space.

The set $P(H)$ of closed linear subspaces of the Hilbert space H form a σ-lattice under the set theoretical ordering. That is, every denumerable subset of $P(H)$ has a least upper bound (called union, denoted by $\vee$) and a greatest lower bound (called meet, denoted by $\wedge$). (In fact $P(H)$ is a complete lattice.) This lattice has a minimal element — the zero subspace — and a maximal element — the whole space. Moreover, there is a unique orthocomplement of each $M \in P(H)$, denoted in the sequel by $M^\perp$. We write $M \perp N$ if M is contained in $N^\perp$, in other words, if M is orthogonal to N.

Let H and H' be Hilbert spaces. A map $u: P(H) \to P(H')$ will be called a *σ-orthohomomorphism* if it preserves σ-meets and orthocomplements (consequently, it preserves σ-unions, maximal and minimal elements as well). A *σ-orthoisomorphism* is a σ-orthohomomorphism which is one-to-one and onto. The following facts are known and easily verifiable.

Proposition 1. *$u: P(H) \to P(H')$ is a σ-orthohomomorphism if and only if*

(i) $u\left(\bigwedge_{n=1}^{\infty} M_n\right) = \bigwedge_{n=1}^{\infty} u(M_n)$ for $M_n \in P(H)$,

(ii) $u(H) = H'$,

(iii) $u(M) \perp u(N)$ *and* $u(M \vee N) = u(M) \vee u(N)$ *for* $M, N \in P(H)$, $M \perp N$.

Proposition 2. *A σ-orthohomomorphism u is injective if and only if $u(M) = 0$ implies $M = 0$.*

We shall use the notation $[x]$ for the subspace generated by the element x of a Hilbert space.

Proposition 3. *Let $u: P(H) \to P(H')$ be a σ-orthohomomorphism. Then* $\dim u([x]) = \dim u([y])$ *for all non-zero $x, y \in H$.*

Proof. One knows that for all $M, N \in P(H')$ satisfying $M \wedge N = 0$,

$$\dim((M \vee N) \wedge N^\perp) = \dim M.$$

Now, let $x, y \in H$ be non-zero vectors, $x \neq y$. Then

$$[x+y] \wedge [x] = [x+y] \wedge [y] = [y] \wedge [x] = 0,$$

$$[x+y] \vee [x] = [x+y] \vee [y] = [y] \vee [x]$$

and the same relations hold for the images by u. Hence, according to the previous remark, we have

$$\dim u([x]) = \dim u([x+y]) = \dim u([y]).$$

Proposition 4. *A σ-orthohomomorphism between Hilbert lattices is necessarily injective.*

Proof. Suppose that a σ-orthohomomorphism is not injective. Then there is non-zero subspace and even a one-dimensional subspace whose image is zero. Consequently, by Proposition 3, the image of any one-dimensional subspace is zero, hence all images are zero, which is a contradiction: the image of the whole space must be the whole space.

Corollary. *If there is a σ-orthohomomorphism from $P(H)$ into $P(H')$, then there is a finite or countably infinite number r such that* $\dim H' = r \cdot \dim H$.

Proof. If e_n $(n=1, 2, \ldots)$ is an orthogonal basis in H, then $u([e_n])$ are pairwise orthogonal subspaces spanning H'. r is the dimension of $u([x])$ for an arbitrary non-zero $x \in H$.

III. Tensor products of Hilbert lattices

Definition 1. Let H_1, H_2 and H be Hilbert spaces, all complex or all real. $(P(H);\, u_1, u_2)$ is called a *tensor product* of $P(H_1)$ and $P(H_2)$ if

(i) $u_i\colon P(H_i) \to P(H)$ is a σ-orthohomomorphism $(i = 1, 2)$,

(ii) $$\bigvee_{n=1}^{\infty} \bigvee_{m=1}^{\infty} \left(u_1(M_1^n) \wedge u_2(M_2^m)\right) = \left(\bigvee_{n=1}^{\infty} u_1(M_1^n)\right) \wedge \left(\bigvee_{m=1}^{\infty} u_2(M_2^m)\right)$$

for any pairwise orthogonal elements M_1^n of $P(H_1)$ and any pairwise orthogonal elements M_2^m of $P(H_2)$,

(iii) $u_1(P(H_1))$ and $u_2(P(H_2))$ generate $P(H)$, that is the smallest orthocomplemented subspace lattice containing both $u_1(P(H_1))$ and $u_2(P(H_2))$ is $P(H)$.

Definition 2. Let $(P(H); u_1, u_2)$ and $(P(H'); u_1', u_2')$ be tensor products of $P(H_1)$ and $P(H_2)$. We say that $(P(H'); u_1', u_2')$ is *subordinated* to $(P(H); u_1, u_2)$ if there is a σ-orthohomomorphism $u\colon P(H) \to P(H')$ such that $u'_i = u \circ u_i$ $(i=1, 2)$. If $(P(H); u_1, u_2)$ is also subordinated to $(P(H'); u'_1, u'_2)$ then the two tensor products are said to be *equivalent.*

Notice the trivial facts that u in Definition 2 is necessarily surjective and it is unique. Indeed, the image of u is an orthocomplemented sublattice of $P(H')$ and it contains a subset — the image of u_1' and of u_2' — generating $P(H')$. Furthermore, if there were two σ-orthohomomorphisms defining the same subordination, they would coincide on a subset — on the image of u_1 and of u_2 — generating $P(H)$, hence they would be equal. By the same reasons, equivalent tensor products are related by σ-orthoisomorphisms.

Subordination is a quasi-ordering on the tensor products of two given Hilbert lattices. After identification of equivalent tensor products the subordination will be an ordering. Our main task is to examine this ordered set. The notations will be as in Definition 1.

Proposition 5. *The only possible subordination between tensor products of Hilbert lattices is equivalence.*

Proof. A σ-orthohomomorphism establishing a subordination is necessarily surjective and also injective by Proposition 4, hence it is a σ-orthoisomorphism.

Proposition 6. *Let $M_2 \in P(H_2)$, $M_2 \neq 0$ be fixed. Then the map f_{1,M_2} from $P(H_1)$ into $P(u_2(M_2))$ defined by*

$$f_{1,M_2}(M_1) = u_1(M_1) \wedge u_2(M_2) \quad (M_1 \in P(H_1))$$

is a σ-orthohomomorphism. The same is true for the map f_{2,M_1} defined similarly for a fixed non-zero element M_1 of $P(H_1)$.

Proof. We show that f_{1,M_2} satisfies conditions (i)—(iii) of Proposition 1. Conditions (i), (ii) are trivially fullfilled. Let now $M_1, N_1 \in P(H_1)$, $M_1 \perp N_1$, and write $f = f_{1,M_2}$. Then $u_1(M_1) \perp u_1(N_1)$ and so $f(M_1) \perp f(N_1)$ as well. Furthermore,

$$\begin{aligned} f(M_1 \vee N_1) &= u_1(M_1 \vee N_1) \wedge u_2(M_2) = \big(u_1(M_1) \vee u_1(N_1)\big) \wedge u_2(M_2) = \\ &= \big(u_1(M_1) \wedge u_2(M_2)\big) \vee \big(u_1(N_1) \vee u_2(M_2)\big) = \\ &= f(M_1) \vee f(N_1), \end{aligned}$$

where we used condition (ii) of Definition 1.

Proposition 7. $u_1(M_1) \wedge u_2(M_2) = 0$ *if and only if either* $M_1 = 0$ *or* $M_2 = 0$.

Proof. f_{1,M_2} of Proposition 6 is injective by Proposition 4. Thus, by Proposition 2, for fixed $M_2 \neq 0$

$$u_1(M_1) \wedge u_2(M_2) = 0 \text{ if and only if } M_1 = 0,$$

and a similar relation holds for a fixed $M_1 \neq 0$.

Proposition 8. $\dim\big(u_1([x_1]) \wedge u_2([x_2])\big)$ *is the same for all* $0 \neq x_1 \in H_1$, $0 \neq x_2 \in H_2$.

Proof. Let us fix $x_2 \in H_2$, $x_2 \neq 0$. Then

$$\dim\big(f_{1,[x_2]}([x_1])\big) = \dim\big(u_1([x_1]) \wedge u_2([x_2])\big)$$

is independent of x_1 (Proposition 3). Similarly, it is independent of x_2.

Proposition 9. *Let e_1^n $(n=1, 2, \ldots)$ and e_2^m $(m=1, 2, \ldots)$ be maximal orthogonal systems in H_1 and in H_2, respectively. Then*

$$u([e_1^n])\wedge u_2([e_2^m]) \qquad (n, m = 1, 2, \ldots)$$

are pairwise orthogonal subspaces which span H.

Proof. They are orthogonal because $u_1([e_1^n])\perp u_1([e_1^{n'}])$ and $u_2([e_2^m])\perp u_2([e_2^{m'}])$ if $n\neq n'$ and $m\neq m'$. Their span is

$$\bigvee_{n=1}^{\infty}\bigvee_{m=1}^{\infty}\left(u_1([e_1^n])\wedge u_2([e_2^m])\right)$$

which equals H by condition (ii) in Definition 1 and the equalities $u_1(H_1)=u_2(H_2)=H$.

Corollary. *There is a finite or countably infinite number r such that* $\dim H = r\cdot\dim H_1\cdot\dim H_2$.

Indeed, $r=\dim\left(u_1([x_1])\wedge u_2([x_2])\right)$ for non-zero $x_1\in H_1$, $x_2\in H_2$.

Now we impose a further condition on tensor products. We expect, roughly speaking, that the image of u_1 and of u_2 fill $P(H)$ the possible fullest. It follows from Proposition 9 that the image, by the map f_{1,M_2}, of a one-dimensional subspace is one-dimensional if and only if $r=1$ and M_2 is one-dimensional. Hence f_{1,M_2} can be surjective only in that case. Now, our requirement of a maximality reads as follows:

Condition of fullness. The σ-orthohomomorphisms $f_{1,[x_2]}$ and $f_{2,[x_1]}$ are surjective for all non-zero $x_2\in H_2$, $x_1\in H_1$.

At this point we introduce a new notation. If K is a complex Hilbert space, $\bar{K}$ denotes its conjugate Hilbert space, that is a Hilbert space whose elements can be canonically identified with the elements of K such that if $\bar{x}$ and $\bar{y}$ in $\bar{K}$ correspond to x and y in K, then $\bar{x}+\bar{y}$ corresponds to $x+y$, $\bar{\lambda}\bar{x}$ corresponds to λx and $\langle\bar{x}, \bar{y}\rangle = \langle y, x\rangle$, where λ is an arbitrary complex number and $\langle , \rangle$ denotes the inner product both in $\bar{K}$ and in K. If K is real, $\bar{K}=K$.

Theorem 1. *Let H_1 and H_2 be Hilbert spaces,* $\dim H_1\geqq 3$, $\dim H_2\geqq 3$. *If the Hilbert spaces are complex, then there exist exactly two (non-equivalent) tensor products of $P(H_1)$ and $P(H_2)$ satisfying the condition of fullness. They are given by*

(i) $H=H_1\otimes H_2$, $u_1(M_1)=M_1\otimes H_2$, $u_2(M_2)=H_1\otimes M_2$;

(ii) $H=\bar{H}_1\otimes H_2$, $u_1(M_1)=\bar{M}_1\otimes H_2$, $u_2(M_2)=\bar{H}_1\otimes M_2$,

where $\otimes$ denotes the usual tensor products of Hilbert spaces.

If the Hilbert spaces are real, there is only one tensor product of $P(H_1)$ and $P(H_2)$ satisfying the condition of fullness. It can be obtained from the above formulae, taking the case (i).

Proof. Only the complex case will be considered, it reflects the real case as well.

Let us choose a vector of norm one in each one-dimensional subspace of H_1 and H_2. Let us denote their set by H_1^0 and H_2^0, respectively.

Let $r_2 \in H_2^0$ be fixed. Now the σ-orthohomomorphism $f_{1,[r_2]}$ is surjective by hypothesis and it is injective by nature, so it is a σ-orthoisomorphism. Hence, by a theorem of E. P. Wigner ([3], pp. 166—169) there exists a unitary or antiunitary map $U_1^{(r_2)}: H_1 \to u_2([r_2])$, determined up to a scalar factor, such that

$$[U_1^{(r_2)} r_1] = u_1([r_1]) \wedge u_2([r_2]) \tag{2}$$

for all $r_1 \in H_1^0$. In the same way, for all $r_1 \in H_1^0$ one finds a unitary or antiunitary map $U_2^{(r_1)}: H_2 \to u_1([r_1])$ such that

$$[U_2^{(r_1)} r_2] = u_1([r_1]) \wedge u_2([r_2])$$

for all $r_2 \in H_2^0$. As a consequence, we are given a map ϑ from $H_1^0 \times H_2^0$ into the complex unit circle such that

$$U_1^{(r_2)} r_1 = \vartheta(r_1, r_2) U_2^{(r_1)} r_2$$

for all $r_1 \in H_1^0$ and $r_2 \in H_2^0$.

Our first aim is to show that ϑ is a product of two maps, one from H_1^0 and the other from H_2^0:

Let $r_i, s_i, t_i \in H_i^0$ $(i=1, 2)$ and $t_i = \lambda(t_i)(r_i + s_i)$, where $\lambda(t_i)$ is an appropriate complex number. We shall write

$$\lambda(t_1)^{(r_2)} = \begin{cases} \lambda(t_1) & \text{if} \quad U_1^{(r_2)} \text{ is unitary,} \\ \overline{\lambda(t_1)} & \text{if} \quad U_1^{(r_2)} \text{ is antiunitary,} \end{cases}$$

and similarly for all other possible choices of indices and representatives. Now we have:

$$U_1^{(t_2)} t_1 = \lambda(t_1)^{(t_2)} \{U_1^{(t_2)} r_1 + U_1^{(t_2)} s_1\} =$$

$$= \lambda(t_1)^{(t_2)} \{\vartheta(r_1, t_2) \lambda(t_2)^{(r_1)} [U_2^{(r_1)} r_2 + U_2^{(r_1)} s_2] + \vartheta(s_1, t_2) \lambda(t_2)^{(s_1)} [U_2^{(s_1)} r_2 + U_2^{(s_1)} s_2]\} =$$

$$= \lambda(t_1)^{(t_2)} \{\vartheta(r_1, t_2) \lambda(t_2)^{(r_1)} [\overline{\vartheta(r_1, r_2)} U_1^{(r_2)} r_1 + \overline{\vartheta(r_1, s_2)} U_1^{(s_2)} r_1] +$$

$$+ \vartheta(s_1, t_2) \lambda(t_2)^{(s_1)} [\overline{\vartheta(s_1, r_2)} U_1^{(r_2)} s_1 + \overline{\vartheta(s_1, s_2)} U_1^{(s_2)} s_1]\}.$$

On the other hand,

$$U_1^{(t_2)} t_1 = \vartheta(t_1, t_2) U_2^{(t_1)} t_2 =$$

$$= \vartheta(t_1, t_2) \lambda(t_2)^{(t_1)} \{\overline{\vartheta(t_1, r_2)} \lambda(t_1)^{(r_2)} [U_1^{(r_2)} r_1 + U_1^{(r_2)} s_1] + \overline{\vartheta(t_1, s_2)} \lambda(t_1)^{(s_2)} [U_1^{(s_2)} r_1 + U_1^{(s_2)} s_1]\}.$$

If $r_1 \neq s_1$ and $r_2 \neq s_2$ then $U_1^{(r_2)} r_1$, $U_1^{(r_2)} s_1$, $U_1^{(s_2)} r_1$ and $U_1^{(s_2)} s_1$ are linearly independent. Indeed, arbitrary two of them are linearly independent and the first two ones generate a subspace whose intersection with the subspace generated by the second two ones consists of zero only.

As a consequence, the two expressions for $U_1^{(t_2)}t_1$ can be equal only if all the corresponding coefficients are equal. In the second equality the coefficient of $U_1^{(r_2)}r_1$ resp. $U_1^{(s_2)}r_1$ equals that of $U_1^{(r_2)}s_1$ resp. $U_1^{(s_2)}s_1$. The same relation must hold in the first equality, whence we obtain

$$\vartheta(r_1, r_2)\vartheta(s_1, s_2) = \vartheta(r_1, s_2)\vartheta(s_1, r_2)$$

for all $r_1, s_1 \in H_1^0$ and $r_2, s_2 \in H_2^0$. It follows that $\vartheta(r_1, r_2) = \varphi_1(r_1)\varphi_2(r_2)$ for some function φ_1 on H_1^0 resp. φ_2 on H_2^0.

Consequently, the unitary or antiunitary maps, in the sense of Wigner's theorem, can be chosen so that

$$U_1^{(r_2)}r_1 = U_2^{(r_1)}r_2 \quad \text{for all} \quad r_1 \in H_1^0, \quad r_2 \in H_2^0.$$

Now we can assert that $U_1^{(r_2)}$ resp. $U_2^{(r_1)}$ are either unitary or antiunitary for all r_2 resp. for all r_1. We have this result from the equalities written for $U_1^{(t_2)}t_1$ taking $\vartheta = 1$.

It is now possible to define a map $U_1^{(x_2)}$ resp. $U_2^{(x_1)}$ for all $x_2 \in H_2$ resp. $x_1 \in H_1$. Let $x_2 = \lambda r_2$ where $r_2 \in H_2^0$ and λ is an appropriate complex number.
Then we define

$$U_1^{(x_2)} = \begin{cases} \lambda U_1^{(r_2)} & \text{if} \quad U_2^{(r_1)} \quad \text{is unitary for all} \quad r_1 \in H_1^0, \\ \bar{\lambda} U_1^{(r_2)} & \text{if} \quad U_2^{(r_1)} \quad \text{is antiunitary for all} \quad r_1 \in H_1^0. \end{cases}$$

A similar definition is made for $U_2^{(x_1)}$.

As a consequence of these definitions, we have a map $b: H_1 \times H_2 \to H$ such that

$$b(x_1, x_2) = U_1^{(x_2)}x_1 = U_2^{(x_1)}x_2 \quad (x_1 \in H_1,\ x_2 \in H_2) \tag{3}$$

and b is bilinear, or sesquilinear with respect to the first or to the second variable, or conjugate bilinear, according to the unitary or antiunitary nature of the $U_1^{(r_2)}$'s and $U_2^{(r_1)}$'s.

Consider the case when b is bilinear. Then there is a unique densely defined linear map F: $H_1 \otimes H_2 \to H$ such that

$$F(x_1 \otimes x_2) = b(x_1, x_2). \tag{4}$$

If e_1^n $(n=1, 2, \ldots)$ and e_2^m $(m=1, 2, \ldots)$ are maximal orthogonal systems in H_1 and in H_2 respectively, then one knows that $e_1^n \otimes e_2^m$ $(n, m=1, 2, \ldots)$ is a maximal orthogonal system in $H_1 \otimes H_2$. By Proposition 9, by (2) and (3), $b(e_1^n, e_2^m)$ $(n, m=1, 2, \ldots)$ is a maximal orthogonal system in H. Thus F can be extended to a unitary map. From (4) one deduces that

$$[x_1] \otimes [x_2] = [x_1 \otimes x_2] = F^{-1}(u_1([x_1]) \wedge u_2([x_2]))$$

for all $x_1 \in H_1$, $x_2 \in H_2$, and it follows by condition 3 in Definition 1 that

$$F^{-1}(u_1(M_1)) = M_1 \otimes H_2 \quad \text{for all} \quad M_1 \in P(H_1),$$
$$F^{-1}(u_2(M_2)) = H_1 \otimes M_2 \quad \text{for all} \quad M_2 \in P(H_2)$$

which establishes an equivalence between the investigated lattice tensor product and the tensor product of the form (i) of Theorem 1.

If b is sesquilinear, we obtain $\overline{H}_1 \otimes H_2$ or $H_1 \otimes \overline{H}_2$. If b is conjugate bilinear, we arrive at $\overline{H}_1 \otimes \overline{H}_2$. There is a canonical antiunitary map between $\overline{H}_1 \otimes H_2$ and $H_1 \otimes \overline{H}_2$ as well as between $H_1 \otimes H_2$ and $\overline{H}_1 \otimes \overline{H}_2$, which are easily seen to establish an equivalence between the corresponding lattice tensor products.

On the contrary, the lattice tensor products corresponding to $H_1 \otimes H_2$ and to $\overline{H}_1 \otimes H_2$ are not equivalent. To see this, assume that there is a σ-orthoisomorphism $u: P(H_1 \otimes H_2) \to P(\overline{H}_1 \otimes H_2)$ such that

$$u(M_1 \otimes H_2) = \overline{M}_1 \otimes H_2 \quad \text{for all} \quad M_1 \in P(H_1),$$
$$u(H_1 \otimes M_2) = \overline{H}_1 \otimes M_2 \quad \text{for all} \quad M_2 \in P(H_2).$$

One knows that $(M_1 \otimes H_2) \wedge (H_1 \otimes M_2) = M_1 \otimes M_2$, thus

$$u(M_1 \otimes M_2) = \overline{M}_1 \otimes M_2 \quad \text{for all} \quad M_1 \in P(H_1), \quad M_2 \in P(H_2)$$

because u preserves meet. This implies that there is a unitary or antiunitary map $U: H_1 \otimes H_2 \to \overline{H}_1 \otimes H_2$ and a map τ from $H_1 \times H_2$ into the complex unit circle such that

$$U(x_1 \otimes x_2) = \tau(x_1, x_2)\bar{x}_1 \otimes x_2$$

for all $x_1 \in H_1$, $x_2 \in H_2$. It is routine to check that this can hold only for $\dim H_1 = \dim H_2 = 1$.

To end this section, let us observe that we can define the tensor product of finitely many Hilbert lattices as well by an easy generalization of Definition 1. It is given explicitely in Discussion in a more general context. Propositions 5, 6, 7, 8 and 9 can be stated and the condition of fullness can be defined in an obviously generalized manner for the case of finitely many Hilbert lattices. Then we have the following result.

Let m be a fixed natural number and let H_i $(i=1, 2, \dots, m)$ be Hilbert spaces. Take an integer s, $0 \leqq s \leqq m$, and let C_s^m be the set of all combinations of order s of $1, \dots, m$. We write for $M_i \in P(H_i)$ and for $p_s \in C_s^m$

$$M_i^{p_s} = \begin{cases} \overline{M}_i & \text{if} \quad i \in p_s \\ M_i & \text{if} \quad i \notin p_s; \end{cases}$$

$[m/2]$ will denote the integral part of $m/2$.

Theorem 2. *Let H_i be Hilbert spaces,* $\dim H_i \geqq 3$ $(i=1, 2, \dots, m)$. *If the Hilbert spaces are complex then there exist exactly 2^{m-1} different (non-equivalent) tensor products of $P(H_i)$ satisfying the condition of fullness. They are given by*

$$H = \bigotimes_{i=1}^{m} H_i^{p_s} \qquad (p_s \in \check{C}_s^m,\ s = 0, 1, \dots, [m/2], \text{ if } m \text{ is even, } m/2 \in p_{m/2}),$$

$$u_i(M_i) = H_1^{p_s} \otimes H_2^{p_s} \otimes \dots \otimes H_{i-1}^{p_s} \otimes M_i^{p_s} \otimes H_{i+1}^{p_s} \otimes \dots \otimes H_m^{p_s} \quad \textit{for all} \quad M_i \in P(H_i)$$

$$(i = 1, 2, \dots, m).$$

If the Hilbert spaces are real, there is only one tensor product satisfying the condition of fullness. It can be obtained from the above formulae putting $s=0$.

IV. Discussion

As it was pointed out in the Introduction we are interested in composed (or product) probability spaces in general. The proper subject of our investigations should be orthomodular σ-lattices; they are general enough to include the basic consepts both of classical and of the most important non-classical probability theory: σ-algebras of subsets as well as Hilbert lattices are orthomodular σ-lattices. The definition and fundamental properties of orthomodular σ-lattices can be found in [3], [4], [5]. We shall consider orthomodular σ-lattices and σ-orthohomomorphisms between them as objects and morphism of a category. For details on categories we refer to [6].

In the sequel $\mathbf{N}$ denotes the set of natural numbers and I is an arbitrarily chosen non-void set.

Definition 3. Let $\mathscr{C}$ be a subcategory of the category of orthomodular σ-lattices. Assume L_i $(i \in I)$ and L are objects of $\mathscr{C}$. Then $(L, (u_i)_{i \in I})$ is a *tensor product* (or *free orthodistributive product)* of the L_i's if

(i) $u_i \colon L_i \to L$ are injections in $\mathscr{C}$ $(i \in I)$;

(ii) $\bigcup_{i \in I} u_i(L_i)$ generates L;

for every finite or countable subset F of I

(iii) $\bigwedge_{i \in F} u_i(a_i) = 0$ for $a_i \in L_i$ if and only if at least one a_i is zero;

(iv) if $(a_i^n)_{n \in \mathbf{N}} \subset L_i$ $(i \in F)$ are subsets consisting of pairwise orthogonal elements, then

$$\bigwedge_{i \in F} \bigvee_{n \in \mathbf{N}} u_i(a_i^n) = \bigvee_{n \in \mathbf{N}^F} \bigwedge_{i \in F} u_i(a_i^{n_i}).$$

The subordination of tensor products can be defined similarly as in Definition 2.

The definition of tensor products is motivated by physical considerations, outlined here briefly. The L_i's ($i \in I$) are logics (see in the Introduction) of given physical systems and we are seeking the logic of the physical system consisting of the given ones. Condition (i) in Definition 3 requires no comment. Condition (ii) expresses that the component physical systems determine somehow the composite system. Condition (iii) reflects that the component systems are independent, that is there are no constraints among them; interactions, however, may occur. Condition (iv) is an expression of the requirement that the events of different components shall be compatible (the notion of compatibility can be found in [3]).

We defined tensor product in the special case of Hilbert lattices only for finitely many objects because we can give a characterization only in that case. Observe that the u_i's are not required to be injections in Definition 1, because they are injections by Proposition 4. Similarly, condition (iii) of Definition 3 is missing from Definition 1 in view of Proposition 6.

Results are available mostly for the full subcategory of Boolean σ-algebras. For instance, if I is finite, then condition (iv) in Definition 3 is void because of the distributivity in L. Then we know that there exists a maximal tensor product in the ordered set of equivalent tensor products ("free Boolean σ-products" [7] p. 177). It is known as well that in the full subcategory of σ-algebras of sets — where condition (iv) of Definition 3 is void for an arbitrary I — there is only one tensor product (up to equivalence) and this is the σ-algebra generated by "cylinders" in the Descartes product of the underlying sets, well-known from measure theory ([7], p. 186).

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Invariant subspaces of von Neumann algebras

By C. PELIGRAD in Bucharest (Romania)

In what follows we denote by H a complex Hilbert space and by $B(H)$ the algebra of all bounded linear operators on H. A vector subspace $K \subset H$ is called semi-closed if there is $t \in B(H)$ such that $K = tH$. An operator $T: D_T \to H$ ($D_T \subset H$) is called semi-closed if its graph $\Gamma_T = \{(x, Tx) | x \in D_T\}$ is a semi-closed subspace of $H \oplus H$. If $B \subset B(H)$, we shall denote by $\mathrm{Lat}(B)$ the set of all closed subspaces of H, invariant for B and by $\mathrm{Lat}_{1/2}(B)$ the set of all semi-closed subspaces of H invariant for B. For $n \in \mathrm{N}$, we denote

$$H^{(n)} = \underbrace{H \oplus H \oplus \ldots \oplus H}_{n \text{ fold}} \quad \text{and} \quad B^{(n)} = \{\underbrace{a \oplus a \oplus \ldots \oplus a}_{n \text{ fold}} | a \in B\}.$$

We say that an algebra $A \subset B(H)$ is *transitive* if it is weakly closed in $B(H)$ and $\mathrm{Lat}(A) = \{(0), H\}$. In [1], [2] (see [8] p. 138) there are given conditions for a transitive algebra to be equal to $B(H)$. An algebra $A \subset B(H)$ is called *strongly transitive* if it is weakly closed in $B(H)$ and $\mathrm{Lat}_{1/2}(A) = \{(0), H\}$.

In [3], C. FOIAŞ has proved that the only strongly transitive algebra is $B(H)$. We say that an algebra $A \subset B(H)$ is reductive if it is weakly closed and $\mathrm{Lat}(A) = \mathrm{Lat}(A^*)$ (where $A^* = \{a^* | a \in A\}$).

In [4], [7] (see [8], p. 167) there are given conditions for a reductive algebra to be a von Neumann algebra. Finally, an algebra $A \subset B(H)$ is called *strongly reductive* if it is weakly closed and $\mathrm{Lat}_{1/2}(A^*) \subset \mathrm{Lat}_{1/2}(A)$. In [9], D. VOICULESCU has proved that if A is a weakly closed algebra with spatial multiplicity $\geqq 3$ and such that $\mathrm{Lat}_{1/2}(A) = \mathrm{Lat}_{1/2}(M)$, where M is the von Neumann algebra generated by A (in particular A is strongly reductive), then $A = M$. Our corollary 1.3 generalizes this result. In § 2 we study reductive algebras which contain von Neumann algebras having property (P) of J. T. SCHWARTZ.

Recall that a von Neumann algebra N has property (P), if for every $t \in B(H)$ the weakly closed convex hull of $\{u^* t u | u \in N, \text{ unitary}\}$ has non-void intersection with the commutant N' of N.

§ 1. Strongly reductive algebras

1.1. Lemma. (See [7]). *Let A and M be weakly closed algebras such that $A \subset M$ and* $\mathrm{Lat}\,(A^{(n)}) = \mathrm{Lat}\,(M^{(n)})$ *for every* $n \in \mathbf{N}$. Then $A = M$.

The following theorem appears in literature in an implicite form:

1.2. Theorem. *Let $A \subset B(H)$ be a reductive algebra. We suppose that for any finite collection $T_1, \ldots, T_n$ of linear oparetors defined on one and the same dense subspace $K \subset H$, the relation $K_{n+1} = \{(x_1, T_1 x, \ldots, T_n x) | x \in K\} \in \mathrm{Lat}\,(A^{(n+1)})$ implies that $K_{n+1} \in \mathrm{Lat}\,(A^{*(n+1)})$. Then A is a von Neumann algebra.*

Proof. We shall prove by induction that the assumption of Lemma 1.1 is also satisfied if M is replaced by the von Neumann algebra M which A generates. In fact, by the reductivity of A we have $\mathrm{Lat}\,(A) = \mathrm{Lat}\,(M)$. Suppose that for $k \leqq n$, $\mathrm{Lat}\,(A^{(k)}) = \mathrm{Lat}\,(M^{(k)})$ and let $L_{n+1} \in \mathrm{Lat}\,(A^{(n+1)})$. Set $L_n = \{(x_1, \ldots, x_{n+1}) \in L_{n+1} | x_1 = 0\}$. As L_n can be considered an element of $\mathrm{Lat}\,(A^{(n)})$ the induction hypothesis implies that $L_n \in \mathrm{Lat}\,(M^{(n+1)})$. Since M is a von Neumann algebra, we have $H^{(n+1)} \ominus L_n \in \mathrm{Lat}\,(M^{(n+1)}) \subset \mathrm{Lat}\,(A^{(n+1)})$. Therefore $L_{n+1} \ominus L_n = (H^{(n+1)} \ominus L_n) \cap L_{n+1} \in \mathrm{Lat}\,(A^{(n+1)})$. If $(x_1, \ldots, x_{n+1}) \in L_{n+1} \ominus L_n$ and $x_1 = 0$, then $x_2 = \ldots = x_{n+1} = 0$.

It follows that there exists a linear subspace $K_0 \subset H$ and linear operators $T_1^0, \ldots, T_n^0$ defined on K_0 such that $L_{n+1} \ominus L_n = \{(x, T_1^0 x \ldots T_1^0 x) | x \in K_0\}$.

For every i $(1 \leqq i \leqq n)$ we define on the dense subspace $K = K_0 + K_0^{\perp}$ the operator T_i in the following way:

$$T_i x = T_i^0 x \quad \text{if} \quad x \in K_i, \quad T_i x = 0 \quad \text{if} \quad x \in K_0^{\perp}$$

It is obvious that

$$L_{n+1} \ominus L_n = \{(x, T_1 x, \ldots, T_n x) | x \in K\} \ominus \big(K_0 \oplus (0) \oplus \ldots \oplus (0)\big)$$

and that $\{x, T_1 x, \ldots, T_n x) | x \in K\} \in \mathrm{Lat}\,(A^{(n+1)})$. By the assumption of the theorem, $\{(x, T_1 x, \ldots, T_n x) | x \in K\} \in \mathrm{Lat}\,(M^{(n+1)})$, and by the reductivity of A, we have $K_0 \oplus \oplus (0) \ldots \oplus (0) \in \mathrm{Lat}\,(M^{(n+1)})$. It follows that $L_{n+1} \ominus L_n \in \mathrm{Lat}\, M^{(n+1)}$. Therefore, $L_{n+1} = (L_{n+1} \ominus L_n) \oplus L_n \in \mathrm{Lat}\, M^{(n+1)}$.

1.3. Corollary. *Let $A \subset B(H)$ be an algebra such that $A^{(2)}$ is strongly reductive Then A is a von Neumann algebra.*

Proof. Let $K \subset H$ be a dense subspace and $T_i \colon K \to H$ $(i = 1, \ldots, n)$ be linear operators such that $K_{n+1} = \{(x, T_1 x, \ldots, T_n x) | x \in K\} \in \mathrm{Lat}\,(A^{(n+1)})$. It is obvious that each T_i $(1 \leqq i \leqq n)$ commutes with A on K.

Let p_{1i} be the projection of $H^{(n+1)}$ onto the first and i th component $(i = 1, \ldots, n)$. Then

$$p_{1i} K_{n+1} = \{(x, T_i x) | x \in K\} \in \mathrm{Lat}_{1/2}(A^{(2)}) \subset \mathrm{Lat}_{1/2}(A^{*(2)}).$$

Therefore each T_i $(i=1, \ldots, n)$ commutes with A^* on K. It follows that $K_{n+1} \in \in \operatorname{Lat}(A^{*(n+1)})$.

By Theorem 1.2 it follows that A is a von Neumann algebra.

§ 2. Reductive algebras

In [1] it is shown that if a reductive algebra A contains a m.a.s.a (maximal abelian self adjoint algebra), then A is a von Neumann algebra. In [2], a more general result is proved: if a reductive algebra A contains an abelian von Neumann algebra with finite commutant, then A is a von Neumann algebra. It is known that the commutative von Neumann algebras (and more generally type I von Neumann algebras) have property (P).

Taking into account Theorem 2.2 below, it is likely that the answer to the following question is in the affirmative:

2.1. Question. If A is a reductive algebra which contains a von Neumann algebra N with property (P) and having finite commutant, then A is a von Neumann algebra.

A partial answer to this question is given by

2.2. Theorem. *Let $A \subset B(H)$ be an algebra such that* 1) *$A^{(2)}$ is reductive;* 2) *$A^{(2)}$ contains a von Neumann algebra $N^{(2)}$ with property (P) and having finite commutant. Then A is a von Neumann algebra.*

In the proof of this theorem we need the following:

2.3. Lemma. *Let $N \subset B(H)$ be a von Neumann algebra with finite commutant. If $N^{(2)}$ has property (P), then every semi-closed, densely defined operator which commutes with N is preclosed.*

Proof. Let $T\colon D_T \to H$ be a semi-closed, densely defined linear operator which commutes with N. Then the linear subspace $\Gamma_T = \{(x, Tx) | x \in D_T\} \subset H^{(2)}$ is a semi-closed subspace, invariant under $N^{(2)}$. Because $N^{(2)}$ has property (P), it follows (cf. [9], Théorème 2) that there exists an operator $Q \in N^{(2)\prime}$ such that $\Gamma_T = Q(H^{(2)}) = Q((\ker Q)^\perp)$. Hence for each $x \in D_T$ there exists a unique $(y_1(x), y_2(x)) \in (\ker Q)^\perp$ such that $(x, Tx) = Q(y_1(x), y_2(x))$. Set $\Delta = \{(x, x) \in H^{(2)} | x \in H\}$.

We now define a linear operator Y on the dense linear subspace $D_Y = (\Delta \cap D_T^{(2)}) + \Delta^\perp \subset H^{(2)}$ as follows:

$$Y(x, x) = (y_1(x), y_2(x)) \quad \text{for} \quad x \in D_T; \qquad Y(z, y) = 0 \quad \text{for} \quad (z, y) \in \Delta^\perp$$

The operator Y is closed. Indeed, let $\{(x_n, x_n) + (z_n, y_n)\}_{n \in \mathbf{N}}$ be such that $(x_n, x_n) + (z_n, y_n) \to (x, x) + (z, y)$ $(x \in H, (z, y) \in \Delta^\perp)$ and $Y((x_n, x_n) + (z_n, y_n)) = (y_1(x_n), y_2(x_n)) \to (u, v) \in (\ker Q)^\perp$ as $n \to \infty$.

Because of the continuity of Q, it follows that $Q(y_1(x_n), y_2(x_n)) \to Q(u, v)$. Therefore $(x_n, Tx_n) \to Q(u, v)$ and $Q(u, v) = Q(y_1(x), y_2(x))$. It follows that $(u, v) = (y_1(x), y_2(x))$ and hence Y is closed. We will show that Y commutes with $N^{(2)}$. Since $Q \in N^{(2)\prime}$ we obtain that $(\ker Q)^\perp$ is invariant under $N^{(2)}$. Now for $x \in D_T$ and $a \in N$ we have

$$a^{(2)}(x, Tx) = (ax, Tax) = Q(y_1(ax), y_2(ax)).$$

On the other hand:

$$a^{(2)}(x, Tx) = a^{(2)} Q(y_1(x), y_2(x)) = Q(ay_1(x), ay_2(x)).$$

By the remark above $(ay_1(x), ay_2(x)) \in (\ker Q)^\perp$, and therefore $(y_1(ax), y_2(ax)) = (ay_1(x), ay_2(x))$.

Since Δ is an invariant subspace under $N^{(2)}$ and $N^{(2)}$ is a von Neumann algebra, it follows that $\Delta^\perp$ is invariant under $N^{(2)}$. Therefore Y commutes with $N^{(2)}$. Let p_2 be the projection of $H^{(2)}$ onto its 2nd component. It is obvious that $Tx = p_2 Q Y(x, x)$. Since $p_2 Q \in N^{(2)\prime}$ and Y is affiliated to $N^{(2)}$ (which is a finite von Neumann algebra), we obtain (cf. [5] and also [6], Theorem XV, p. 119) that $p_2 Q Y$ is preclosed and therefore T is preclosed.

Proof of Theorem 2.2. We shall verify the hypothesis of Theorem 1.2. Let $K \subset H$ be a dense subspace, and $T_1, \dots, T_n$ linear operators defined on K and such that $K_{n+1} = \{(x, T_1 x, \dots, T_n x) | x \in K\} \in \mathrm{Lat}\,(A^{(n+1)})$. As in the proof of Corollary 1.3, it follows that for every i $(1 \leqq i \leqq n)$ the graph $\Gamma_{T_i} = \{(x, T_i x) | x \in K\}$ is semi-closed and therefore the operators T_i, $1 \leqq i \leqq n$, are semi-closed.

By Lemma 2.3 the operators T_i $(1 \leqq i \leqq n)$ are preclosed. Let $\overline{T}_i$ be the closure of T_i $(1 \leqq i \leqq n)$, and $K_0 = \bigcap_{i=1}^{n} D_{\overline{T}_i}$. Obviously, $K \subset K_0$. Since $A^{(2)}$ is reductive, $\overline{T}_i$ commutes with A^*. Set $\Delta_n = \{(x, x, \dots, x) \in H^{(n)} | x \in H\}$ and define the operators T and T_0 on the dense subspaces $(\Delta_n \cap K^{(n)}) + \Delta_n^\perp \subset H^{(n)}$ and $\Delta_n K^{(n)} + \Delta_n^\perp \subset H^{(n)}$ respectively in the following way:

$$T(x, x, \dots, x) = (T_1 x, \dots, T_n x) \quad \text{if} \quad (x, \dots, x) \in \Delta_n \cap K^{(n)},$$

$$T(x_1, \dots, x_n) = 0 \quad \text{if} \quad (x_1, \dots, x_n) \in \Delta_n^\perp$$

and

$$T_0(x, x, \dots, x) = (\overline{T}_1 x, \dots, \overline{T}_n x) \quad \text{if} \quad (x, \dots, x) \in \Delta_n \cap K_0^{(n)},$$

$$T_0(x_1, \dots, x_n) = 0 \quad \text{if} \quad (x_1, \dots, x_n) \in \Delta_n^\perp.$$

Because Δ_n is invariant under $A^{(n)}$, and therefore under $N^{(n)}$, it is easily seen that T and T_0 are closed operators affiliated with the finite von Neumann algebra

$N^{(n)}$. Thus $K \subset K_0$ implies that $T \subset T_0$. According to [5] (see also [6], Theorem XV, p. 119) we obtain that $T = T_0$. By the remark above T_0 commutes with $A^{*(n)}$, and therefore T commutes with $A^{*(n)}$. But this means that $K_{n+1} \in \mathrm{Lat}\,(A^{*(n+1)})$.

Added in proof. We remark that Lemma 2.3. holds without the assumption „$N^{(2)}$ has property (P)", so Theorem 2.2. can be improved: Let $A \subset B(H)$ be an algebra which contains a von Neumann algebra with finite commutant and such that $A^{(2)}$ is reductive. Then A is a von Neumann algebra. Proofs of these improvements will appear elsewhere.

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On a generalization of the concept of orthogonality

By F. SCHIPP in Budapest

To Professor K. Tandori on his 50th birthday

1. Definitions and theorems

Let $(X, \mathscr{A}, \mu)$ be a probability space,

$$\mathscr{A}_0 = \{X, \emptyset\} \subset \mathscr{A}_1 \subset \ldots \subset \mathscr{A}_n \subset \ldots$$

a sequence of sub-σ-algebras of the σ-algebra $\mathscr{A}$, and suppose that $\mathscr{A} = \mathscr{A}_\infty = \bigvee_n \mathscr{A}_n$.

Furthermore, let $\mathrm{N} = \{0, 1, 2, \ldots\}$, $\overline{\mathrm{N}} = \mathrm{N} \cup \{\infty\}$, $L^p(\mathscr{A}_n) = L^p(X, \mathscr{A}_n, \mu)$ $(n \in \mathrm{N}$, $1 \leqq p \leqq \infty)$, and denote by $\|f\|_p$ the $L^p(\mathscr{A})$-norm of the function $f \in L^p(\mathscr{A})$.

Using the notation of [1] we call a mapping $\tau: X \to \overline{\mathrm{N}}$ a *stopping time* relative to the sequence $\mathbf{A} = (\mathscr{A}_n, n \in \mathrm{N})$ if for every $n \in \mathrm{N}$ we have $\{\tau = n\} \in \mathscr{A}_n$.

Denote by $\mathscr{T}$ the set of stopping times relative to $\mathbf{A}$ and for every $\tau \in \mathscr{T}$ introduce the class of sets

$$\mathscr{A}_\tau := \{A \in \mathscr{A}: A \cap \{\tau = n\} \in \mathscr{A}_n \ (\forall n \in \mathrm{N})\}.$$

It is known that $\mathscr{A}_\tau \subset \mathscr{A}$ is a σ-algebra, τ is $\mathscr{A}_\tau$-measurable, and if $\tau = n = \mathrm{const}$ $(n \in \mathrm{N})$ then $\mathscr{A}_\tau$ equals $\mathscr{A}_n$ (see e.g. [1]). Moreover it is clear that for every $\tau, \nu \in \mathscr{T}$ their envelopes $\tau \vee \nu$ and $\tau \wedge \nu$ also belong to $\mathscr{T}$.

For any stopping time $\tau \in \mathscr{T}$ denote by E_τ the condtitional expectation operator relative to $\mathscr{A}_\tau$, in particular E_n $(n \in \overline{\mathrm{N}})$ denotes the conditional expectation operator relative to $\mathscr{A}_n$. It is known that E_∞ is equal to the identity, and for every $\tau \in \mathscr{T}$ we have $I\{\tau = n\} E_\tau = I\{\tau = n\} E_n$.[1)]

Let $\tau_i \in \mathscr{T}$ $(i \in \mathscr{I})$ be a system of stopping times labeled by the elements i of some set $\mathscr{I}$ of indices. Denote $T = (\tau_i, i \in \mathscr{I})$, and let $\Phi = \{\varphi_i, i \in \mathscr{I}\}$ be a system of functions $\varphi_i \in L^2(\mathscr{A})$. The sequence T will be fixed throughout this paper.

[1)] $I(A)$ denotes then indicator function of the set $A \subset X$.

Using these notations we introduce the following generalization of the concept of orthogonality.

Definition. The system Φ is called a *T-orthogonal system* (briefly T-OS) if for every $i, j \in \mathscr{I}$, $i \neq j$

$$E_{\tau_i \vee \tau_j}(\varphi_i \bar{\varphi}_j) = 0. \tag{1}$$

If there exists a system of non empty sets $A_i \in \mathscr{A}_{\tau_i}$ $(i \in \mathscr{I})$ such that

$$E_{\tau_i}(|\varphi|^2) = I(A_i) \quad (i \in \mathscr{I}), \tag{2}$$

then Φ is called a *T-normed system*. Systems which are T-orthogonal and T-normed are called *T-orthogonal systems* (T-ONS).

We note that any system Φ can be made T-normed by multiplication of its elements by appropriate functions. Namely, set

$$A_i = \{E_{\tau_i}(|\varphi_i|^2) \neq 0\} \quad (i \in \mathscr{I}), \tag{3}$$

and $\chi_i = 0$ on $X \setminus A_i$ and $\chi_i = (E_{\tau_i}(|\varphi_i|^2))^{-1/2}$ on A_i. Then χ_i is $\mathscr{A}_{\tau_i}$ measurable, and by

$$E_{\tau_i}(|\chi_i \varphi_i|^2) = |\chi_i|^2 E_{\tau_i}(|\varphi_i|^2) = I(A_i)$$

$\{\chi_i \varphi_i : i \in \mathscr{I}\}$ is a T-normed system.

If $\tau_i = 0$ $(i \in \mathscr{I})$, then $E_{\tau_i \vee \tau_j}(\varphi_i \bar{\varphi}_j) = \int_X \varphi_i \bar{\varphi}_j \, d\mu$ so in this case the above definition reduce to that of usual ONS.

In this note we will prove a generalization of Bessel's identity for T-ONS as follows:

Theorem 1. *Let $\Phi = \{\varphi_i : i \in \mathscr{I}\}$ be a $T = (\tau_i, i \in \mathscr{I})$-ONS, $\mathscr{I}_0$ a finite subset of $\mathscr{I}$, and $\tau \in \mathscr{T}$ a stopping time such that $\tau \leqq \tau_i$ for every $i \in \mathscr{I}$. Then for any function $f \in L^2(\mathscr{A})$ we have*

$$\inf \Big\{E_\tau\big(\big|f - \sum_{i \in \mathscr{I}_0} \lambda_i \varphi_i\big|^2\big) : \lambda_i \in L^2(\mathscr{A}_{\tau_i})\Big\} = E_\tau(|f|^2) - \sum_{i \in \mathscr{I}_0} E_\tau(|E_{\tau_i}(f \bar{\varphi}_i)|^2), \tag{4}$$

and the infimum is attained for $\lambda_i = E_{\tau_i}(f \bar{\varphi}_i)$.

In case $\tau_i = \tau = 0$ $(i \in \mathscr{I})$ this identity reduces to the usual Bessel's identity. (4) immediately implies the following generalization of Bessel's inequality:

Corollary 1. *The set*

$$\mathscr{I}_f = \{i \in \mathscr{I} : E_\tau(f \bar{\varphi}_i) \neq 0\}$$

is at most countable and

(5) $$\sum_{i\in\mathcal{I}_f} E_\tau(|E_{\tau_i}(f\bar\varphi_i)|^2) \leqq E_\tau(|f|^2).$$

Let us now introduce the following generalization of the concepts of Fourier coefficients and Fourier expansion.

Definition. Let $\Phi=\{\varphi_i\colon i\in\mathcal{I}\}$ be a T-ONS. The function $E_{\tau_i}(f\bar\varphi_i)$ $(i\in\mathcal{I})$ is called the i-th *T-Fourier coefficient*, and the series

$$S[f] = \sum_{i\in\mathcal{I}_f} E_{\tau_i}(f\bar\varphi_i)\varphi_i$$

the *T-Fourier series*, of the function f with respect to the system Φ.

The converse of Corollary 1 gives a generalization of Riesz—Fischer-theorem.

Theorem 2. *Let $\Phi=\{\varphi_i\colon i\in\mathcal{I}\}$ be a* T-ONS, *$\mathcal{I}_0=\{i_n\colon n\in\mathrm{N}\}\subset\mathcal{I}$, and $\tau\in\mathcal{T}$ a stopping time with $\tau\leqq\tau_i$ $(i\in\mathcal{I})$. Furthermore, let $\lambda_i\in L^2(\mathcal{A}_{\tau_i})$ be a sequence satisfying the conditions*

$$\lambda_i = 0 \quad (i\in\mathcal{I}\setminus\mathcal{I}_0), \qquad \sum_{i\in\mathcal{I}_0}\int_{A_i}|\lambda_i|^2\,d\mu<\infty.$$

Then there exists a (unique) function $f\in L^2(\mathcal{A})$ such that

(6) a) $I(A_i)\lambda_i = E_{\tau_i}(f\bar\varphi_i)$ $(i\in\mathcal{I})$, b) $\lim\limits_{N\to\infty} E_\tau(|f-\sum\limits_{n=0}^{N}\lambda_{i_n}\varphi_{i_n}|^2) = 0.$

The following concept is a generalization of the completeness relative to the space $L^2(\mathcal{A})$.

Definition. *A* system $\Phi=\{\varphi_i\colon i\in\mathcal{I}\}\subset L^2(\mathcal{A})$ is *T-complete* (relative to the space $L^2(\mathcal{A})$) if $f\in L^2(\mathcal{A})$, and $E_{\tau_i}(f\varphi_i)=0$ $(i\in\mathcal{I})$ imply $f=0$.

From Theorems 1 and 2, and Corollary 1 it follows in a simple way the following

Corollary 2. *If Φ is an T-complete* T-ONS, *then for every function $f\in L^2(\mathcal{A})$ the relations*

(7) a) $\lim\limits_{N\to\infty} E_\tau(|f-\sum\limits_{n=0}^{N}E_{\tau_{i_n}}(f\bar\varphi_{i_n})\varphi_{i_n}|^2) = 0,$ b) $E_\tau(|f|^2) = \sum\limits_{i\in\mathcal{I}_f} E_\tau(|E_{\tau_i}(f\bar\varphi_i)|^2)$

hold; here $\mathcal{I}_f=\{i_n\colon n\in\mathrm{N}\}$.

Statement a) means that the Fourier series of any function $f\in L^2(\mathcal{A})$ with respect to an T-complete T-ONS converges in the "norm" $\|\cdot\|_{(\mathcal{A}_\tau,2)}=[E_\tau(|\cdot|)^2]^{1/2}$ to the function f.

2. Proofs

First we recall some properties of the conditional expectations which we are going to use.

Let $\tau, \nu\in\mathcal{T}$. Then

(8) $$\{\tau<\nu\},\quad \{\tau=\nu\},\quad \{\tau\leqq\nu\}\in\mathcal{A}_\tau\cap\mathcal{A}_\nu$$

and if $\tau\leqq\nu$, then

(9) $$\mathcal{A}_\tau\subset\mathcal{A}_\nu \quad\text{and}\quad E_\tau\circ E_\nu = E_\nu\circ E_\tau = E_\tau,$$

where $\circ$ denotes the composition of functions. Moreover it is known that if λ is $\mathcal{A}_\tau$-measurable and if f and $\lambda f\in L^1(\mathcal{A})$ then

(10) $$E_\tau(\lambda f) = \lambda E_\tau f.$$

We note that this equations also holds for any $\mathcal{A}$-measurable $f\colon X\to[0,\infty]$ and $\mathcal{A}_\tau$-measurable $\lambda\colon X\to[0,\infty]$. (See e.g. [1], p. 7 and 9.)

It follows from the above properties that for arbitrary stopping times $\tau, \nu\in\mathcal{T}$

(11) $$E_\tau\circ E_\nu = E_\nu\circ E_\tau = E_{\tau\wedge\nu}.$$

Namely, let $f\in L^1(\mathcal{A})$. Then by (9)

(12) $$E_{\tau\wedge\nu}f = I\{\tau<\nu\}E_\tau f + I\{\tau\geqq\nu\}E_\nu f = I\{\tau<\nu\}E_\tau(E_{\tau\vee\nu}f) + I\{\tau\geqq\nu\}E_{\tau\vee\nu}(E_\nu f).$$

Since by (10)

$$I\{\tau<\nu\}E_{\tau\vee\nu}f = I\{\tau<\nu\}E_\nu f = E_\nu(I\{\tau<\nu\}f)$$

and similarly for every function $g\in L^1(\mathcal{A})$

$$I\{\tau\geqq\nu\}E_{\tau\vee\nu}g = E_\tau(I\{\tau\geqq\nu\}g),$$

therefore from (12) by (10) we have

$$E_{\tau\wedge\nu}f = E_\tau(I\{\tau<\nu\}E_{\tau\vee\nu}f) + I\{\tau\geqq\nu\}E_{\tau\vee\nu}(E_\nu f) =$$

$$= E_\tau(E_\nu(I\{\tau<\nu\}f)) + E_\tau(I\{\tau\geqq\nu\}E_\nu f) = (E_\tau\circ E_\nu)(I\{\tau<\nu\}f + I\{\tau\geqq\nu\}f) =$$

$$= (E_\tau\circ E_\nu)f.$$

Similarly, we get $E_\nu\circ E_\tau = E_{\tau\wedge\nu}$.

Further on we often refer to the following

Remark. *Let* $\varphi_i\in L^2(\mathcal{A})$, $A_i=\{E_{\tau_i}(|\varphi_i|^2)\neq 0\}$ $(i\in\mathcal{I})$. *Then*

(13) $$I(A_i)\varphi_i = \varphi_i \quad (i\in\mathcal{I}).$$

From the definition of the conditional expectation and from that of sets A_i it follows that

$$0 = \int_{X \setminus A_i} E_{\tau_i}(|\varphi_i|^2)\, d\mu = \int_{X \setminus A_i} |\varphi_i|^2\, d\mu;$$

thus we have $I(X \setminus A_i)\ \varphi_i = 0$. Hence we obtain

$$\varphi_i = I(X \setminus A_i)\varphi_i + I(A_i)\varphi_i = I(A_i)\varphi_i \quad (i \in \mathscr{I}),$$

and our statement is proved.

Proof of Theorem 1. Let $\lambda_i \in L^2(\mathscr{A}_{\tau_i})$ $(i \in \mathscr{I}_0)$. Then by (10), taking into account the T-normedness of the system Φ, we have

$$\int_X |\lambda_i \varphi_i|^2\, d\mu = E_0\big(E_{\tau_i}(|\lambda_i \varphi_i|^2)\big) = E_0\big(|\lambda_i|^2 E_{\tau_i}(|\varphi_i|^2)\big) = E_0\big(|\lambda_i|^2 I(A_i)\big) < \infty.$$

Hence it follows that for $\lambda_i \in L^2(\mathscr{A}_{\tau_i})$ we have $\lambda_i \varphi_i \in L^2(\mathscr{A})$. Using the additivity of E_τ we obtain

$$\delta := E_\tau\big(|f - \sum_{i \in \mathscr{I}_0} \lambda_i \varphi_i|^2\big) = E_\tau(|f|^2) - \sum_{i \in \mathscr{I}_0} E_\tau(\bar{\lambda}_i f \bar{\varphi}_i + \lambda_i \bar{f} \varphi_i) + \sum_{i, j \in \mathscr{I}_0} E_\tau(\lambda_i \bar{\lambda}_j \varphi_i \bar{\varphi}_j).$$

Since by (11), (10), (1), and (2)

$$E_\tau(\lambda_i \bar{\lambda}_j \varphi_i \bar{\varphi}_j) = (E_\tau \circ E_{\tau_i \vee \tau_j})(\lambda_i \bar{\lambda}_j \varphi_i \bar{\varphi}_j) = E_\tau\big(\lambda_i \bar{\lambda}_j E_{\tau_i \vee \tau_j}(\varphi_i \bar{\varphi}_j)\big) = E_\tau\big(\lambda_i \bar{\lambda}_j I(A_i) \delta_{ij}\big),$$

and

$$E_\tau(\bar{\lambda}_i f \bar{\varphi}_i) = (E_\tau \circ E_{\tau_i})(\bar{\lambda}_i f \bar{\varphi}_i) = E_\tau\big(\bar{\lambda}_i E_{\tau_i}(f \bar{\varphi}_i)\big)$$

therefore by (13) δ can be expressed as follows:

$$\delta = E_\tau(|f|^2) - \sum_{i \in \mathscr{I}_0} E_\tau\big(\lambda_i I(A_i) \overline{E_{\tau_i}(f \bar{\varphi}_i)} + \bar{\lambda}_i I(A_i) E_{\tau_i}(f \bar{\varphi}_i)\big) +$$

$$+ \sum_{i \in \mathscr{I}_0} E_\tau(I(A_i)|\lambda_i|^2) = E_\tau(|f|^2) + \sum_{i \in \mathscr{I}_0} E_\tau\big(|E_{\tau_i}(f \bar{\varphi}_i) - \lambda_i I(A_i)|^2\big) - \sum_{i \in \mathscr{I}_0} E_\tau\big(|E_{\tau_i}(f \bar{\varphi}_i)|^2\big).$$

Hence it is obvious that δ is minimal if $\lambda_i = E_{\tau_i}(f \bar{\varphi}_i)$ and we have (4) as asserted.

Proof of Theorem 2. Let $S_N = \sum_{n=0}^{N} \lambda_{i_n} \varphi_{i_n}$, $\mathrm{N} \ni M > N$. Then by the above Remark we have $S_N \in L^2(\mathscr{A})$ $(N \in \mathrm{N})$, and from (1) and (2) we obtain

$$E_\tau(|S_M - S_N|^2) = \sum_{N < k, l \leq M} E_\tau(\lambda_{i_k} \bar{\lambda}_{i_l} \varphi_{i_k} \bar{\varphi}_{i_l}) =$$

(14)

$$= \sum_{N < k, l \leq M} E_\tau\big(\lambda_{i_k} \bar{\lambda}_{i_l} E_{\tau_{i_k} \vee \tau_{i_l}}(\varphi_{i_k} \bar{\varphi}_{i_l})\big) = \sum_{N < k \leq M} E_\tau\big(|\lambda_{i_k}|^2 I(A_{i_k})\big).$$

Hence

$$\|S_N - S_M\|_2^2 = \sum_{N < k \leq M} \int_{A_{i_k}} |\lambda_{i_k}|^2\, d\mu \to 0 \quad (M, N \to \infty).$$

From the last inequality it is clear that there exists a sequence $(N_k, k\in\mathbf{N})$ such that S_{N_k} is convergent μ-a.e. and $f:=\lim_{k\to\infty} S_{N_k}\in L^2(\mathscr{A})$. Applying Fatou's theorem for the conditional expectation (see e.g. [1], p. 9) and taking the limit from (14) as $M\to\infty$ we obtain

$$E_\tau(|f-S_N|^2) \leqq \varrho_N := \sum_{k=N+1}^{\infty} E_\tau(|\lambda_{i_k}|^2 I(A_{i_k})).$$

Since $\sum_n \int_{A_{i_n}} |\lambda_{i_n}|^2\, d\mu<\infty$ implies $\varrho_N\to 0$ μ-a.e. as $N\to\infty$, the validity of statement (6) b) for f follows.

From Hölder's inequality for the conditional expectation (see e.g. [1], p. 10) we get for any function $g\in L^2(\mathscr{A})$

$$|E_\tau(f\bar g)-E_\tau(S_N\bar g)| \leqq [E_\tau(|f-S_N|^2)]^{1/2}[E_\tau(|g|^2)]^{1/2}\to 0 \tag{15}$$

μ-a.e. as $N\to\infty$.

If $g=\bar\chi_i\varphi_i$, where $\chi_i\in L^\infty(\mathscr{A}_{\tau_i})$, we have

$$E_\tau(S_N\bar g) = \sum_{k=0}^{N} (E_\tau\circ E_{\tau_{i_k}\vee\tau_i})(\lambda_{i_k}\varphi_{i_k}\bar g) = \sum_{k=0}^{N} E_\tau\big(\lambda_{i_k}\chi_i E_{\tau_{i_k}\vee\tau_i}(\varphi_{i_k}\bar\varphi_i)\big) =$$

$$= \begin{cases} E_\tau\big(\lambda_i\chi_i I(A_i)\big) & (i\in\{i_0, \dots, i_N\}),\\ 0 & (i\notin\{i_0, \dots, i_N\})\end{cases}$$

and similarly

$$E_\tau(f\bar g) = E_\tau\big(\chi_i E_{\tau_i}(f\bar\varphi_i)\big).$$

Hence using (15) we obtain that

$$E_\tau\big(\chi_i(E_{\tau_i}(f\bar\varphi_i)-\lambda_i I(A_i))\big) = 0 \quad (i\in\mathscr{I}),$$

whence choosing $\chi_i=\operatorname{sgn}(E_{\tau_i}(f\bar\varphi_i)-\lambda_i I(A_i))$[2] $(i\in\mathscr{I})$ we get the desired equality (6) a).

3. Examples

In this section we indicate some examples for the concepts introduced before.

1° Let μ be the Lebesgue measure on $X=[0, 1)$ and $\mathscr{A}$ the class of Lebesgue measurable subsets of X. For every $n\in\mathbf{N}$ define $\mathscr{A}_n$ to be the σ-algebra generated by the dyadic intervals $[k2^{-n}, (k+1)2^{-n}]$ $(k=0, 1, 2, \dots, 2^{n-1})$. Then for any $x\in [k2^{-n}, (k+1)2^{-n}]$ and $f\in L^1(\mathscr{A})$

$$(E_n f)(x) = 2^{-n}\int_{k2^{-n}}^{(k+1)2^{-n}} f\, d\mu. \tag{16}$$

[2]) $\operatorname{sgn} z=\bar z/|z|$ $(z\neq 0)$, and $\operatorname{sgn} 0=0$.

Denote by $\Phi=\{\varphi_n: n\in\mathbf{P}=\mathrm{N}\setminus\{0\}\}$ the Rademacher system, i.e. define $\varphi_n(x)=$ $=\varphi_1(2^{n-1}x)$ $(n\in\mathbf{P})$, where

$$\varphi_1(x)=\begin{cases} 1 & (0\leqq x<1/2) \\ -1 & (1/2\leqq x<1)\end{cases}, \quad \text{and} \quad \varphi_1(x+1)=\varphi_1(x) \quad (x\in\mathbf{R}).$$

Then

(17) $$\text{a) } \varphi_n\in L^\infty(\mathscr{A}_n), \quad \text{b) } E_{n-1}(\varphi_n)=0 \quad (n\in\mathbf{P});$$

thus Φ is an T-ONS, where $T=(n-1, n\in\mathbf{P})$.

Equality (16) easily implies that the T-Fourier series of a function $f\in L^1(\mathscr{A})$ with respect to the system Φ is the same as the Haar-Fourier series of f.

In this example the Rademacher system can be replaced by any system $\Phi=$ $=\{\varphi_n: n\in\mathrm{N}\}\subset L^2(X, \mathscr{A}, \mu)$ consisting of independent functions having the property

$$\int_X \varphi_n \, d\mu = 0 \quad (\mathrm{n}\in\mathrm{N}).$$

2° It can be shown [4] that the polynomials $P_k(\cdot, \omega)$ which play an important role in papers [5] and [6] can also be obtained by T-Fourier expansions with respect to an appropriate system.

3° For a fixed $N\in\mathbf{P}$ denote by $\mathscr{A}_n$ $(n=0, 1, \dots, N)$ the class of Lebesgue measurable 2^{-N+n}-periodic subsets of the set $X=[0, 1)$, and define $\varphi_n(x)=\exp(2\pi i 2^{N-n}x)$ $(x\in X, n=0, 1, \dots, N)$. It is not hard to prove that $\Phi=\{\varphi_n: n=0, 1, \dots, \mathrm{N}\}$ is a $T=(n-1, n\in\{0, 1, \dots, N\})$-ONS, see [4].

Further examples can be found in [3] and [4].

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On orthogonal trigonometric polynomials

By I. SZALAY in Szeged

Dedicated to Professor K. Tandori on his 50th birthday

1. Let Φ, $\mathcal{T}$, $\mathcal{P}$ respectively denote the set of all orthonormal systems $\varphi = \{\varphi_n(x)\}_1^\infty$, the set of orthonormal systems $T=\{T_n(x)\}_1^\infty$ consisting of trigonometric polynomials, and the set of orthonormal systems $P=\{P_n(x)\}_1^\infty$ consisting of algebraic polynomials, on the interval $[0, 2\pi]$.

For any given set $\mathcal{H}$ of orthonormal systems $H=\{H_n(x)\}_1^\infty$ on $[0, 2\pi]$, a sequence $\{a_n\}_1^\infty$ of real numbers is said to be a *convergence sequence over* $\mathcal{H}$ if for each $H\in\mathcal{H}$ the series $\sum_{n=1}^\infty a_n H_n(x)$ converges almost everywhere in $[0, 2\pi]$.

For any sequence $\{a_n\}_1^\infty$ of real numbers we define

$$\|\{a_n\}_M^N, \mathcal{H}\|_p = \sup_{H\in\mathcal{H}} \Big(\int_0^{2\pi} \sup_{M\leqq i<j<N} \Big|\sum_{n=i+1}^{j} a_n H_n(x)\Big|^p dx\Big)^{1/p}$$

$(1\leqq p\leqq 2;\ 0\leqq M<N\leqq\infty)$.

It can be shown that

$$\lim_{N\to\infty} \|\{a_n\}_0^N, \mathcal{H}\|_p = \|\{a_n\}_0^\infty, \mathcal{H}\|_p. \tag{1}$$

In [3] TANDORI proved the following

Theorem A. *The sequence $\{a_n\}_1^\infty$ is a convergence sequence over Φ if and only if $\|\{a_n\}_1^\infty, \Phi\|_p<\infty$ $(1\leqq p\leqq 2)$.*

In [1] LEINDLER proved two deep approximation theorems for orthonormal polynomials and using these he proved, roughly saying, that if a divergence theorem can be sated for a general orthogonal series then there exists a series of orthogonal polynomials for which the same divergence phenomenon holds.

In the present paper we prove the analogues of Leindler's theorems for orthogonal trigonometric polynomials.

Theorem 1. Let $\varphi\in\Phi$. *For any sequence $\{\varepsilon_k\}_1^\infty$ of positive numbers and any sequence $\{N_k\}_0^\infty$ of integers $(0=N_0<N_1<\dots)$ there exist a system $T\in\mathcal{T}$ and a sequence*

$\{G_k\}_1^\infty$ of measurable subsets of $[0, 2\pi]$ *such that for any $x \in \mathrm{C}G_k$ and n satisfying $N_{k-1} < n \leqq N_k$ we have*

$$|\varphi_n(x) - (-1)^{j_k(x)} T_n(x)| \leqq \varepsilon_k \qquad (j_k(x) = 0 \text{ or } 1), \tag{2}$$

$$\mu(G_k) \leqq \varepsilon_k \qquad (k = 1, 2, \ldots), \tag{3}$$

and

$$\max_{x \in [0, 2\pi]} |T_n(x)| \leqq \sqrt{2}\left(\sup_{0 < x < 2\pi} |\varphi_n(x)| + 1\right). \tag{4}$$

Theorem 2. *Let $\varphi \in \Phi$. Let $\{a_n\}_1^\infty$ be a sequence of real numbers and $\{b_n\}_1^\infty$ a non-decreasing sequence of positive numbers. Suppose that $\{\mathscr{H}_k\}_1^\infty$ is a sequence of measurable subsets of* $[0, 2\pi]$, *$\{N_k\}_0^\infty$ is a given sequence of integers $(0 = N_0 < N_1 < \ldots)$, and ε is a given positive number. If $\mu\,(\overline{\lim_k}\, \mathscr{H}_k) = 2\pi$ if and for each $x \in \mathscr{H}_k$ there is a pair of integers $\nu_k(x)$, $\mu_k(x)$ such that $N_k \leqq \nu_k(x) < \mu_k(x) \leqq N_{k+1}$ and*

$$\left|\sum_{n_k = \nu(x)+1}^{\mu_k(x)} a_n \varphi_n(x)\right| \geqq b_k, \tag{5}$$

then there exists a $T \in \mathscr{T}$ such that the inequality

$$\left|\sum_{n = \nu_k(x)+1}^{\mu_k(x)} a_n T_n(x)\right| \geqq (1 - \varepsilon) b_k \tag{6}$$

holds for infinitely many k almost everywhere in $[0, 2\pi]$. *If the system φ is uniformly bounded then the system T can also be chosen uniformly bounded.*

Using Theorems 1 and 2 and results of Tandori we prove the following theorems.

Theorem 3. *If $1 \leqq p \leqq 2$ then the inequalities*

$$\|\{a_n\}_0^\infty, \mathscr{T}\|_p \leqq \|\{a_n\}\|_0^\infty, \Phi\|_p \leqq 2^{1-\frac{1}{p}} \|\{a_n\}_0^\infty, \mathscr{T}\|_p \tag{7}$$

hold.

Theorem 4. *The sequence $\{a_n\}_1^\infty$ is a convergence sequence over $\mathscr{T}$ if and only if $\|\{a_n\}_0^\infty, \mathscr{T}\|_p < \infty$ $(1 \leqq p \leqq 2)$.*

Finally, from Theorem A and Theorems 3, 4 we get immediately

Theorem 5. *A sequence $\{a_n\}_1^\infty$ of reals is a convergence sequence over Φ if and only if it is a convergence sequence over $\mathscr{T}$.*

We remark that Theorems 3—5 hold true for $\mathscr{P}$ instead of $\mathscr{T}$, too.

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2. We require the following lemmas. The proof of our first lemma is completely similar to that of one of Leindler's lemmas ([1], p. 26) so we omit its proof.

Lemma 1. *Let* $\{\psi_n(x)\}_1^\infty$ *be a system of measurable and bounded functions, and* $\{N_k\}_0^\infty$ *a given sequence of integers* $(0=N_0<N_1<\dots)$. *If for each* k $(k=1, 2, \dots)$ *the system* $\{\psi_n(x)\}_{N_{k-1}+1}^{N_k}$ *is orthonormal in the interval* $[0, 2\pi]$ *then for every given sequence* $\{\varepsilon_k\}_1^\infty$ *of positive numbers there exist a system* $T\in\mathcal{T}$ *and a sequence* $\{E_k\}_1^\infty$ *measurable subsets of* $[0, 2\pi]$ *such that for any* $x\in \complement E_k$ *and* $N_{k-1}<n\leqq N_k$

$$|\psi_n(x)-(-1)^{j_k(x)}|T_n(x) < \varepsilon_k \qquad (j_k(x) = 0 \text{ or } 1), \tag{8}$$

$$\mu(E_k) \leqq \varepsilon_k \qquad (k = 1, 2, \dots) \tag{9}$$

and

$$\max_{x\in[0,2\pi]} |T_n(x)| \leqq \sqrt{2}\big(\sup_{0<x<2\pi} |\psi_n(x)|+1\big) \qquad (n = 1, 2, \dots). \tag{10}$$

Lemma 2. (Leindler [1], p. 33) *Let* $\varphi\in\Phi$. *For every given sequence* $\{\varepsilon_k\}_1^\infty$ *of positive numbers and any sequence* $\{N_k\}_0^\infty$ *of integers* $(0=N_0<N_1<\dots)$, *there exist a normed system* $\{\psi_n(x)\}_1^\infty$ *of measurable and bounded functions and a sequence* $\{\mathcal{H}_k\}_1^\infty$ *of measurable subsets of* $[0, 2\pi]$ *such that, for every* k $(k=1, 2, \dots)$,

$$\int_0^{2\pi} \psi_n(x)\psi_m(x)\,dx = 0 \qquad (N_{k-1} < n < m \leqq N_k), \tag{11}$$

$$|\varphi_n(x)-\psi_n(x)| < \varepsilon_k \quad \textit{on} \quad \complement\mathcal{H}_k \qquad (N_{k-1} < n \leqq N_k), \tag{12}$$

$$\mu(\mathcal{H}_k) \leqq \varepsilon_k, \tag{13}$$

$$\sup_{0<x<2\pi} |\psi_n(x)| \leqq \sup_{0<x<2\pi} |\varphi_n(x)|. \tag{14}$$

On the basis of a lemma of Tandori [3], p. 222, and by (1) we get

Lemma 3. *If* $1\leqq p\leqq 2$ *and* $1\leqq N\leqq\infty$ *then*

$$\varrho\,\|\{a_n\}_0^N, \Phi\|_2 \leqq \|\{a_n\}_0^N, \Phi\|_p \leqq \|\{a_n\}_0^N, \Phi\|_2$$

where ϱ *is a positive absolute constant.*

Lemma 4. (Tandori [3], p. 220) *If* $1\leqq M<N<\infty$ *then*

$$\|\{a_n\}_0^N, \Phi\|_2 \leqq \|\{a_n\}_0^{M+1}, \Phi\|_2+\|\{a_n\}_M^N, \Phi\|_2.$$

A partial result in the proof of Tandori's theorem ([2], p. 146) we use as

Lemma 5. *Let* $\{N_k\}_0^\infty$ *be a given sequence of integers* $(0=N_0<N_1<\dots)$. *If*

$$\sum_{k=0}^\infty \|\{a_n\}_{N_k}^{N_{k+1}+1}, \Phi\|_2^2 = \infty, \tag{15}$$

then there exist a system $\varphi\in\Phi$ *and a sequence* $\{E_k\}_1^\infty$ *of stochastically independent subsets of* $[0, 2\pi]$ *(every* E_k *is a union of intervals of finite number) such that for each* k

$$\mu(E_k) \geqq \alpha\|\{a_n\}_{N_k}^{N_{k+1}+1}, \Phi\|_2^2 \quad (\alpha \textit{ is a positive constant}), \tag{16}$$

furthermore there exist integers $\nu_k=\nu_k(x)$, $\mu_k=\mu_k(x)$ *such that* $N_k\leqq\nu_k(x)<\mu_k(x)\leqq$

$\leqq N_{k+1}$ *and*

(17) $$\left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)} a_n\varphi_n(x)\right| \geqq 1 \quad \text{for} \quad x\in E_k.$$

3. Proof of Theorem 1. Applying Lemma 2 to the system φ and the sequences $\left\{\frac{\varepsilon_k}{2}\right\}_1^\infty$ and $\{N_k\}_0^\infty$ we get that there exist a normed system of measurable and bounded functions ψ and a sequence $\{\mathscr{H}_k\}_1^\infty$ of measurable sets such that (11) is fulfilled. By (12) and (13) we have that $\mu(\mathscr{H}_k)<\frac{\varepsilon_k}{2}$ and if $x\in\mathbf{C}\mathscr{H}_k$ then $|\varphi_n(x)-\psi_n(x)|<\frac{\varepsilon_k}{2}$ $(N_{k-1}<n\leqq N_k;\ k=1,2,\dots)$. Now applying Lemma 1 with the system ψ and the above mentioned sequences we obtain that there exist a system T and a sequence $\{E_k\}_1^\infty$ of measurable sets such that $\mu(E_k)<\frac{\varepsilon_k}{2}$ (see (9)) and if $x\in\mathbf{C}E_k$ then

$$|\psi_n(x)-(-1)^{j_k(x)}T_n(x)|<\frac{\varepsilon_k}{2} \qquad (N_{k-1}<n\leqq N_k;\ k=1,2,\dots;\ j_k(x) \text{ as in (8)}).$$

Let $G_k=\mathscr{H}_k\cup E_k$ $(k=1,2,\dots)$. Collecting the above facts we immediately obtain (2) and (3). By (14) and (10) we have (4), too.

4. Proof of Theorem 2. Let

(18) $$\varepsilon_k=\varepsilon/[2^k(N_k-N_{k-1})\max\{1,|a_{N_{k-1}+1}|,\dots,|a_{N_k}|\}].$$

Applying Theorem 1 to the system φ and the sequence $\{\varepsilon_k\}_1^\infty$ and $\{N_k\}_0^\infty$ we get that there exist a system T and a sequence of measurable sets $\{G_k\}_1^\infty$ such that (2) and (3) are fulfilled.

Let us choose a natural number ν such that $2^{-(\nu+1)}\leqq b_1$. If $k\geqq\nu$ and $x\in\mathscr{H}_k-G_{k+1}$ then using (2), (5) and (18) we obtain

$$b_k\leqq\left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)}(-1)^{j_{k+1}(x)}a_nT_n(x)\right|+\left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)}a_n\big(\varphi_n(x)-(-1)^{j_{k+1}(x)}T_n(x)\big)\right|\leqq$$

$$\leqq\left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)}a_nT_n(x)\right|+\big(\mu_k(x)-\nu_k(x)\big)\varepsilon_{k+1}\max\{|a_{\nu_k}(x)+1|,\dots,|a_{\mu_k}(x)|\}\leqq$$

$$\leqq\left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)}a_nT_n(x)\right|+\varepsilon b_k,$$

thus (6) holds.

It remains to show that inequality (6) is fulfilled almost everywhere in $[0,2\pi]$, that is, to show that almost all x belong to the sets $\mathscr{H}_k-G_{k+1}$ for infinite many indexes k. Thus it is sufficient to prove that $\mu(\overline{\lim_k}\, G_k)=0$. But this follows from

$$\mu(\overline{\lim_k}\, G_k)\leqq\mu\left(\bigcup_{k=m}^\infty G_k\right)\leqq\sum_{k=m}^\infty\mu(G_k)\leqq\sum_{k=m}^\infty\varepsilon_k\leqq\sum_{k=m}^\infty(\varepsilon/2^k)=\varepsilon/2^{m-1}.$$

If the system φ is uniformly bounded, then by (4) so is the system T too.

5. Proof of Theorem 3. First of all we remark that since $\mathscr{T}\subset\Phi$, the first inequality (7) is evident. Furthermore by (1) it is enough to show that for every integer $N>0$ the inequality

$$\|\{a_n\}_0^N, \Phi\|_p \leqq 2^{1-\frac{1}{p}} \|\{a_n\}_0^N, \mathscr{T}\|_p \tag{19}$$

holds.

Let $\varphi=\{\varphi_n(x)\}_1^\infty\in\Phi$ be an arbitrary but fixed system. As the functions $\varphi_n(x)$ are square-integrable so are the function $\delta_N(x)=\max\limits_{0\leqq i<j<N}\left|\sum\limits_{n=i+1}^{j} a_n\varphi_n(x)\right|$. Therefore, for an arbitrary $\varepsilon(>0)$ there exists a $\delta'(>0)$ such that for every measurable set G with $\mu(G)<\delta'$ we have

$$\int_G \delta_N^p(x)\,dx \leqq (\varepsilon/2)^p \qquad (1\leqq p\leqq 2). \tag{20}$$

For any i and j $(0\leqq i<j<N)$ let

$$\delta=\delta(i,j,N,\delta',\varepsilon,\{a_n\})=\min\{\delta'/2^{i+j}, \varepsilon/(8N\pi \max_{1\leqq n<N}|a_n|)\}. \tag{21}$$

By Theorem 1 there exist a system $\{T_n^{(i,j)}(x)\}_1^\infty\in\mathscr{T}$ and a measurable set $G^{(i,j)}$ such that if $i<n\leqq j$ then for any $x\in\complement G^{(i,j)}$

$$|\varphi_n(x)-(-1)^{j(x)}T_n^{(i,j)}(x)|\leqq\delta \qquad (j(x)=0 \text{ or } 1) \tag{22}$$

and

$$\mu(G^{(i,j)})\leqq\delta. \tag{23}$$

If $x\in\complement G^{(i,j)}$ then by (22) we get

$$\left|\sum_{n=i+1}^{j} a_n\varphi_n(x)\right| \leqq \left|\sum_{n=i+1}^{j} a_n T_n^{(i,j)}(x)\right| + \delta\sum_{n=i+1}^{j}|a_n|$$

and considering (21) we have

$$\left|\sum_{n=i+1}^{j} a_n\varphi_n(x)\right|^p \leqq 2^{p-1}\left|\sum_{n=i+1}^{j} a_n T_n^{(i,j)}(x)\right|^p + (\varepsilon/4\pi)^p, \tag{24}$$

where $1\leqq p\leqq 2$.

Set $G_N=\bigcup\limits_{0\leqq i<j<N} G^{(i,j)}$. Using (21) and (23) we get

$$\mu(G_N)\leqq\sum_{0\leqq i<j<N}\mu(G^{(i,j)})\leqq\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta'/2^{i+j}=\delta'. \tag{25}$$

If $x\in\complement G_N$, by (24), we have

$$\delta_N^p(x)\leqq 2^{p-1}\max_{0\leqq i<j<N}\left|\sum_{n=i+1}^{j} a_n T_n^{(i,j)}(x)\right|^p + (\varepsilon/4\pi)^p \qquad (1\leqq p\leqq 2)$$

and considering (20) and (25) we get

$$\int_0^{2\pi} \delta_N^p(x)\,dx = \Big(\int_{\complement G_N} + \int_{G_N}\Big)\delta_N^p(x)\,dx \leqq$$

$$\leqq 2^{p-1}\int_0^{2\pi} \max_{0\leqq i<j<N}\left|\sum_{n=i+1}^{j} a_n T_n^{(i,j)}(x)\right|^p dx + \varepsilon^p \leqq 2^{p-1}\|\{a_n\}_0^N, \mathcal{T}\|_p^p + \varepsilon^p$$

$$(1 \leqq p \leqq 2).$$

Hence we can see that

$$\sup_{\varphi\in\Phi}\left(\int_0^{2\pi} \max_{0\leqq i<j<N}\left|\sum_{n=i+1}^{j} a_n\varphi_n(x)\right|^p dx\right)^{1/p} \leqq 2^{1-\frac{1}{p}}\|\{a_n\}_0^N, \mathcal{T}\|_p + \varepsilon.$$

Considering that ε was arbitrary small we have (19), thus our proof is complete.

6. Proof of Theorem 4. By Theorem A and Theorem 3 the sufficiency is obvious.

To prove the necessity we assume $\|\{a_n\}_0^\infty, \mathcal{T}\|_p = \infty$. Applying Theorem 3 and Lemma 3 we have $\|\{a_n\}_0^\infty, \Phi\|_2 = \infty$.

By (1) and Lemma 4 we obtain that $\lim\limits_{N\to\infty} \|\{a_n\}_M^N, \Phi\|_2 = \infty$ for any M; thus there exists a sequence $\{N_k\}_0^\infty$ $(0=N_0<N_1<\ldots)$ such that $\|\{a_n\}_{N_k}^{N_{k+1}+1}, \Phi\|_2 \geqq 1$ for every k.

For the sequence $\{N_k\}_0^\infty$ we can apply Lemma 5 and we get a system $\varphi\in\Phi$ and a sequence $\{E_k\}_1^\infty$ of stochastically independent sets such that (16) is fulfilled and if $x\in E_k$ then (17) holds.

Considering (15), (16), and applying the second Borel-Cantelli lemma we get $\mu(\overline{\lim\limits_k}\, E_k) = 2\pi$.

Taking the system φ, the sequences $\{a_n\}_1^\infty$, $b_n\equiv 1$ $(n=1,2,\ldots)$, and choosing $\varepsilon=1/2$, it follows from Theorem 2 that there exists a system $T\in\mathcal{T}$ such that the inequality $\left|\sum_{n=\nu_k(x)+1}^{\mu_k(x)} a_n T_n(x)\right| \geqq \frac{1}{2}$ holds for infinitely many k, almost everywhere in $[0, 2\pi]$. This implies that the series $\sum_{n=1}^{\infty} a_n T_n(x)$ diverges almost everywhere in $[0, 2\pi]$.

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On non-localizable measure spaces

By J. SZŰCS in Szeged

0. Introduction

In [S] I. E. SEGAL writes that "it is easily seen that a localizable space is strongly equivalent to its completion, but the question appears to be open in general ... it seems plausible that the answer is negative". In the present paper we establish Segal's conjecture by exhibiting an example of a non-localizable measure space whose completion is localizable.

1. Some notions from measure theory

As a general reference we use the fundamental paper [S]. For the sake of the reader's convenience and since we slightly change some of the definitions of [S] we compile here the measure theoretic notions used in section 2.

A conditional σ-ring $\mathscr{R}$ of subsets of a set R is a ring of subsets of R that is closed under countable intersection.[1] A measure space is a triple $M=(R, \mathscr{R}, r)$ which consists of a set R, a conditional σ-ring $\mathscr{R}$ of subsets of R and a finite non-negative real valued function r defined on $\mathscr{R}$ such that if $\{E_n\}$ is a sequence of mutually disjoint elements of $\mathscr{R}$ and $E=\bigcup_n E_n$ belongs to $\mathscr{R}$ then $r(E)=\sum_n r(E_n)$. A subset F of R is said to be measurable if $F\cap E\in\mathscr{R}$ for all $E\in\mathscr{R}$. The measure of a measurable set F is the least upper bound of the values of r on all those elements of $\mathscr{R}$ which are subsets of F.[2] The measure as a set function extends r and is denoted also by r. A subset F of R is called a null set if it is measurable and $r(F)=0$. The measure

[1]) This definition is different from but equivalent to Definition 2.1 in [S].

[2]) This definition is different from Definition 2.1 in [S]. However, we are going to show that it is basicly the same. To this end define $\mathscr{R}'$ as the collection of all sets E for which there exists a sequence $\{E_n\}$ of elements of $\mathscr{R}$ such that $E=\bigcup_n E_n$ and $\sum_n r(E_n)<+\infty$. It is immediate that $\mathscr{R}'$ is a conditional σ-ring. For each element E of $\mathscr{R}'$ let $r'(E)=\sup\{r(F)\colon F\in\mathscr{R},\ F\subseteq E\}$. Then $(R, \mathscr{R}', r')$ is a measure space in the sense of Definition 2.1 of [S]. Furthermore, it is easy to verify that the measurable sets and their measures are the same in $(R, \mathscr{R}', r')$ as in $(R, \mathscr{R}, r)$.

space M is complete, by definition, if any subset of a null set in $\mathscr{R}$ is a null set in $\mathscr{R}$, or, equivalently, if any subset of a null set is a null set. The completion of M is, by definition, the measure space $M_c=(R, \mathscr{R}_c, r_c)$ where $\mathscr{R}_c$ consists of all those subsets E of R for which there exists an element F of $\mathscr{R}$ such that $(E-F)\cup(F-E)$ is contained in some element of $\mathscr{R}$ of measure zero, and then $r_c(E)=r(F)$. It can be shown by routine methods that in any measure space the collection of all measurable sets is a complemented Boolean σ-ring on which the measure is countably additive. The measure ring $\mathscr{M}$ of a measure space $M=(R, \mathscr{R}, r)$ is, by definition, the quotient of the ring of all measurable sets in M by the ideal of null sets.[3] It is clear that $\mathscr{M}$ is a complemented Boolean σ-ring. Two measure spaces are said to be strongly equivalent if their measure rings are isomorphic as Boolean rings. A measure space is called localizable if its measure ring is complete, i.e., every subset of it has a least upper bound.

2. A non-localizable space whose completion is localizable

Let I be any non-empty set and $R=I\times[0, 1]$ where $[0, 1]$ denotes the unit interval of reals. Let $\mathscr{R}$ be the collection of all subsets of R of the form[4]

$$E = [\bigcup_{i\in J} \{i\}\times E_i]\cup[\bigcup_{x\in T} \bigcup_{i\in K(x)} \{(i, x)\}]$$

where T is a finite subset of $[0, 1]$; the set J is a finite subset of I; for each $x\in T$ the set $K(x)$ is a co-countable subset of I (i.e. $I-K(x)$ is countable) and E_i is a Lebesgue-measurable subset of $[0, 1]$ for all $i\in J$. It is easy to verify that $\mathscr{R}$ is a conditional σ-ring. The equality $r(E)=\sum_{i\in J}\text{mes}\,(E_i)$ defines a countably additive finite positive measure on $\mathscr{R}$.

We are going to show that if the cardinality of I is greater than that of the continuum then the measure space $M=(R, \mathscr{R}, r)$ is non-localizable. To this end let L be a subset of I such that card $L=$card $(I-L)$. Denote by θ the canonical mapping of the set of all measurable sets of M onto the measure ring $\mathscr{M}$ of M. We show that $\mathscr{N}=\{\theta(\{i\}\times[0, 1]): i\in L\}$ does not have a least upper bound in $\mathscr{M}$. Suppose the contrary and denote by F an arbitrary but fixed representative of the l.u.b. of $\mathscr{N}$.

Let H be the set of all those i's ($i\in I$) for which there exists an $x\in[0,1]$ such that $(i, x)\in F$ and $(j, x)\in F$ for only countably many j's ($j\in I$). It is obvious that card H does not exceed the cardinality of the continuum. Hence on account of the assumption

[3]) Differently from Definition 2.4 in [S] we do not define any measure on $\mathscr{M}$.

[4]) We write $\{i\}$ for the singleton which contains i and (i, x) for the ordered pair whose first and second elements are i and x, respectively.

on the cardinality of L there exists an element r of L such that for every $x \in [0, 1]$ the relation $(r, x) \in F$ implies that $(i, x) \in F$ for non-countably many i's $(i \in I)$. Let X be the set of those x's, $x \in [0, 1]$, for which $(i, x) \in F$ for non-countably many i's Then $(\{r\} \times [0, 1]) \cap F \subseteq \{r\} \times X$. We are going to show that X has Lebesgue measure zero which contradicts the fact that $\theta(F)$ is an upper bound of $\mathcal{N}$.

The measurability of F implies that X equals the set of those x's, $x \in [0, 1]$ for which $(i, x) \in F$ for co-countably many i's in I. On account of the assumption on the cardinality of $I-L$ there exists an element s of $I-L$ such that $\{s\} \times X \subseteq F$. On the other hand, $r(F \cap (\{s\} \times [0, 1]))=0$ because $s \in I-L$ and F is a representative of the *least* upper bound of $\mathcal{N}$. This implies that X has Lebesgue measure zero.

The completion of M is strongly equivalent to the direct sum $\bigoplus_{i \in I} [0, 1]$ ($[0, 1]$ with Lebesgue measure) which is localizable.[5] Since M is not localizable this implies that M cannot be strongly equivalent to its completion.

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[5]) Concerning the notion of direct sum and the fact that any direct sum of finite measure spaces is localizable we refer the reader to [S]. However, one can see directly that the completion under consideration is localizable.

Zur Eindeutigkeit des Holomorphs bestimmter Ringe

Von LUTZ WARLICH in Worms (BR Deutschland)

L. RÉDEI hat in seiner Arbeit [4] in Analogie zu dem Holomorph einer Gruppe den Begriff der Holomorphe eines Ringes eingeführt, indem er Paare von Abbildungen eines Ringes in sich definierte, die bezüglich der Ringstruktur vergleichbare Eigenschaften besitzen wie die Automorphismen einer Gruppe bezüglich der Gruppenstruktur. Für gruppentheoretische Sätze über die mit dem Begriff „Holomorph einer Gruppe" eng zusammenhängenden Begriffe „charakteristische Untergruppe" und „vollständige Gruppe" lassen sich nun ringtheoretische Entsprechungen ableiten. Während das Holomorph einer Gruppe eindeutig bestimmt ist, kann ein Ring mehrere Holomorphe haben. Die Frage, unter welchen Bedingungen ein Ring genau ein Holomorph hat, wurde mehrfach untersucht, unter anderem von VAN LEEUWEN [1], [2], RÉDEI [4], WEINERT/EILAUER [5]. In seiner Arbeit [3] beweist G. POLLÁK ein hinreichendes Kriterium für die Einzigkeit des Holomorphs eines Ringes. Eine von ihm angegebene Folgerung aus diesem Kriterium soll in dieser Arbeit widerlegt werden. Es sei R ein Ring, ferner $E(R^+)$ der Endomorphismenring des Moduls R^+, in dem die Verknüpfungen $+$, $\circ$ durch

$$(\alpha+\beta)a = \alpha a+\beta a \quad \text{und} \quad (\alpha\circ\beta)a = \alpha(\beta a) \quad (a\in R,\ \alpha,\beta\in E(R^+))$$

definiert sind. $E_0(R^+)$ sei der zu $E(R^+)$ entgegengesetzte Ring mit den Verknüpfungen

$$a(\alpha+\beta) = a\alpha+a\beta \quad \text{und} \quad a(\alpha\circ\beta) = (a\alpha)\beta \quad (a\in R,\ \alpha,\beta\in E_0(R^+)).$$

Die direkte Summe dieser Ringe

$$\mathfrak{E}(R^+) = E(R^+)\oplus E_0(R^+)$$

heißt voller Doppelendomorphismenring von R^+. Für $a\in R$ und $\alpha=(\alpha_1,\alpha_2)$, $\beta=(\beta_1,\beta_2)\in\mathfrak{E}(R^+)$ gilt also

$$\alpha:\quad a\mapsto(\alpha a, a\alpha) = (\alpha_1 a, \alpha_2 a),$$

$$\alpha+\beta:\quad a\mapsto((\alpha+\beta)a, a(\alpha+\beta)) = (\alpha a+\beta a, a\alpha+a\beta) = (\alpha_1 a+\beta_1 a, \alpha_2 a+\beta_2 a).$$

$$\alpha\circ\beta:\quad a\mapsto((\alpha\circ\beta)a, a(\alpha\circ\beta)) = (\alpha(\beta a), (a\alpha)\beta) = (\alpha_1(\beta_1 a), \beta_2(\alpha_2 a)).$$

Ein Doppelendomorphismus α heißt ein Doppelhomothetismus, falls gilt

$$\alpha(ab) = (\alpha a)b, \quad (ab)\alpha = a(b\alpha), \quad (a\alpha)b = a(\alpha b), \quad (\alpha a)\alpha = \alpha(a\alpha)$$

für alle $a, b \in R$. Zwei Doppelhomothetismen α, β mit

$$(\alpha a)\beta = \alpha(a\beta) \quad \text{und} \quad \beta(a\alpha) = (\beta a)\alpha \quad \text{für alle} \quad a \in R$$

nennt man befreundet. Als Ring befreundeter Doppelhomothetismen bezeichnet man jeden in $\mathfrak{E}(R^+)$ enthaltenen Ring von paarweise befreundeten Doppelhomothetismen. Nach RÉDEI [4] ist jeder Ring von befreundeten Doppelhomothetismen von R in mindestens einem maximalen Ring D befreundeter Doppelhomothetismen enthalten. Als Holomorphe eines Ringes R bezeichnet man die zerfallenden Everettschen Ringerweiterungen von R mit den maximalen Ringen befreundeter Doppelhomothetismen von R. Ist $D * R$ ein Holomorph von R, so sind die Verknüpfungen $+, \circ$ in $D * R$ also definiert durch

$$\left.\begin{aligned} (\alpha, a)+(\beta, b) &= (\alpha+\beta, a+b) \\ (\alpha, a)\circ(\beta, b) &= (\alpha\circ\beta, \beta_2 a+\alpha_1 b+ab) \end{aligned}\right\} \quad \begin{aligned} a, b\in R, \quad \alpha &= (\alpha_1, \alpha_2)\in\mathfrak{E}(R^+), \\ \beta &= (\beta_1, \beta_2)\in\mathfrak{E}(R^+). \end{aligned}$$

Die Einzigkeit des Holomorphs eines Ringes R ist also genau dann gewährleistet, wenn es nur einen maximalen Ring befreundeter Doppelhomothetismen von R gibt.

POLLÁK [3] beweist in seiner Arbeit:

Satz 1. *Hat der Ring R einen charakteristischen Unterring R', der ein einziges Holomorph hat, und ist dabei jeder Homomorphismus von R/R' in den Annullator N von R der Nullhomomorphismus, so hat auch R genau ein Holomorph.*

Dabei heißt ein Unterring L von R *charakteristisch*, falls er unter allen Doppelhomothetismen von R invariant ist. (L ist dann insbesondere ein Ideal, da jedes $a \in R$ einen Doppelhomothetismus induziert.) Nach Pollák folgt aus diesem Satz, daß R ein einziges Holomorph hat, wenn dies für ein Primideal R' von R gilt, weil R' dann charakteristisch (RÉDEI [4]) und R/R' nullteilerfrei ist. Nun soll zunächst gezeigt werden, daß für einen Ring R mit Primideal R' (R' habe genau ein Holomorph) nichttriviale Ringhomomorphismen von R/R' in den Annullator existieren können, daß also für einen solchen Ring im allgemeinen nicht die Voraussetzungen von Satz 1 erfüllt sind. Dazu wähle man einen nullteilerfreien Ring A, der homomorph auf einen Zeroring $Z \neq \{0\}$ mit kommutativem Endomorphismenring $E(Z^+)$ abgebildet werden kann. $\alpha: A \to Z$ sei ein Homomorphismus. Man setze $R = A \oplus Z$. Dann ist Z Primideal und $R/Z (\cong A)$ nullteilerfrei. Ferner hat Z wegen des kommutativen Endomorphismenringes nur ein Holomorph. Der Annullator von R ist Z. Daher gibt es einen nichttrivialen Ringhomomorphismus von R/Z in Z, denn bezeichnet β den Isomorphismus von R/Z auf A, so ist $\nu = \alpha \circ \beta$ ein solcher.

Sei $\mathbf{Z}[x]$ der Polynomring über $\mathbf{Z}$ (dem Ring der ganzen Zahlen). Als Ring A wähle man zum Beispiel den Unterring $x \cdot \mathbf{Z}[x]$ aller Polynome der Form $p(x) = \sum_{i=1}^{n} a_i x^i$ mit $n \in \mathbf{N}$, $a_i \in \mathbf{Z}$. Ferner sei Z der aus der additiven abelschen Gruppe $(\mathbf{Z}, +)$ gebildete Zeroring und $R = x \cdot \mathbf{Z}[x] \oplus Z$. Die Abbildung

$$\alpha\colon x \cdot \mathbf{Z}[x] \to Z,$$

definiert durch

$$\alpha\colon \sum_{i=1}^{n} a_i x^i \to a_1 \qquad (a_i \in \mathbf{Z})$$

ist ein nichttrivialer Ringhomomorphismus von $x \cdot \mathbf{Z}[x]$ in den Annullator Z von R, der wegen $x \cdot \mathbf{Z}[x] \cong R/Z$ einen nichttrivialen Ringhomomorphismus von R/Z in den Annullator Z von R induziert.

Obwohl Satz 1 demzufolge auf den Ring $R = x \cdot \mathbf{Z}[x] \oplus Z$ nicht anwendbar ist, könnte dennoch R genau ein Holomorph haben. Weitere Sätze von POLLÁK [3] ermöglichen eine weitergehende Untersuchung des Ringes R hinsichtlich der Anzahl seiner Holomorphe.

Satz 2. *Sei $R = R_1 \oplus R_2$ und bezeichne N_2 den Annullator von R_2. Genau dann ist R_1 charakteristisch in R, falls nur der triviale Ringhomomorphismus $\nu\colon R_1 \to N_2$ existiert.*

Satz 3. *Hat der Ring R ein Holomorph, so ist jeder direkte Summand in R charakteristisch.*

Da der Ringhomomorphismus $\alpha\colon x \cdot \mathbf{Z}[x] \to Z$ ein nichtrivialer Homomorphismus von $x \cdot \mathbf{Z}[x]$ in den Annullator von Z ist, kann $x \cdot \mathbf{Z}[x]$ wegen Satz 2 nicht charakteristisch in R sein, so daß R nach Satz 3 mindestens zwei Holomorphe besitzt. Zwei nichtbefreundete Doppelhomothetismen von R erhält man folgendermaßen: Für $\alpha'\colon x \cdot \mathbf{Z}[x] \oplus Z \to Z$, definiert durch

$$\alpha'\bigl(p(x), z\bigr) = \bigl(0, \alpha p(x)\bigr),$$

ist $\alpha^* = (\alpha', \alpha')$ ein Doppelhomothetismus von R, wie man leicht nachprüft. Ist π die Projektion von R auf $x \cdot \mathbf{Z}[x]$, also $\pi\bigl(p(x), z\bigr) = \bigl(p(x), 0\bigr)$, so ist auch $\pi^* = (\pi, \pi)$ ein Doppelhomothetismus von R. Für $(2x, 1) \in x \cdot \mathbf{Z}[x] \oplus Z$ ist

$$\alpha^*\bigl((2x, 1)\pi^*\bigr) = \alpha^*(2x, 0) = (0, 2)$$

und

$$\bigl(\alpha^*(2x, 1)\bigr)\pi^* = (0, 2)\pi^* = (0, 0),$$

also sind α^* und π^* nicht befreundet.

In der Arbeit von WEINERT—EILHAUER [5] wird bewiesen: Ist $R=R_1\oplus R_2$ ein Ring, in dem ein Zeroring R_2 als direkter Summand auftritt, so hat R mehrere Holomorphe, wenn dies für R_2 zutrifft.
Der Ring $R=x\cdot Z[x]\oplus Z$ zeigt, daß das hinreichende Kriterium dieses Satzes nicht notwendig ist.

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On nonorthogonal decompositions of certain contractions

By PEI YUAN WU in Bloomington (Indiana, U.S.A.)

SZ.-NAGY and FOIAŞ showed in [4] that a contraction T on a separable Hilbert space H is similar to a unitary operator if and only if its characteristic function $\Theta_T(\lambda)$ has a bounded analytic inverse (see also [5], Ch. IX). In the present paper, we give a generalization of this result. We prove that a contraction T is similar to a direct sum of a unitary operator and a contraction of class $C_{.0}$ if and only if the outer factor of $\Theta_T(\lambda)$ has a bounded analytic inverse. We shall also indicate some interesting consequences.

1. Preliminaries. We only consider non-trivial, complex, separable Hilbert spaces. For completely non-unitary contractions we will use the functional models as developed in [5], Ch. VI.

Let T be a contraction on the Hilbert space H. Denote by $D_T=(1-T^*T)^{1/2}$, $D_{T^*}=(1-TT^*)^{1/2}$ the defect operators and $\mathfrak{D}_T=\overline{D_T H}$, $\mathfrak{D}_{T^*}=\overline{D_{T^*}H}$ the defect spaces of T.

The characteristic function $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$ of T is the purely contractive analytic function from $\mathfrak{D}_T$ to $\mathfrak{D}_{T^*}$ defined by

$$\Theta_T(\lambda) = [-T+\lambda D_{T^*}(1-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T \quad \text{for} \quad |\lambda|<1.$$

If T is completely non-unitary, we will consider T in its functional model, i.e. defined by

$$T^*(u\oplus v) = e^{-it}[u(e^{it})-u(0)]\oplus e^{-it}v(t)$$

on the space

$$H = [H^2(\mathfrak{D}_{T^*})\oplus\overline{\Delta_T L^2(\mathfrak{D}_T)}]\ominus\{\Theta_T u\oplus\Delta_T u:\ u\in H^2(\mathfrak{D}_T)\},$$

where $\Delta_T(t)=[I-\Theta_T(e^{it})^*\Theta_T(e^{it})]^{1/2}$. Let $\Theta_T(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$ be the canonical factorization of $\{\mathfrak{D}_T, \mathfrak{D}_{T^*}, \Theta_T(\lambda)\}$ into the product of its outer factor $\{\mathfrak{D}_T, \mathfrak{F}, \Theta_1(\lambda)\}$ and inner factor $\{\mathfrak{F}, \mathfrak{D}_{T^*}, \Theta_2(\lambda)\}$. Let

$$H_1 = \{\Theta_2 u\oplus v:\ u\in H^2(\mathfrak{F}),\ v\in\overline{\Delta_T L^2(\mathfrak{D}_T)}\}\ominus\{\Theta_T w\oplus\Delta_T w:\ w\in H^2(\mathfrak{D}_T)\}$$

be the induced invariant subspace for T and

$$H_2 = H\ominus H_1 = [H^2(\mathfrak{D}_{T^*})\ominus\Theta_2 H^2(\mathfrak{F})]\oplus\{0\}$$

its orthogonal complement. Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2\end{bmatrix}$ be the triangulation of T corresponding to the decomposition $H=H_1 \oplus H_2$. Recall that a contraction T is of class $C_{\cdot 0}(C_{0\cdot})$ if $T^{*n}h \to 0$ $(T^n h \to 0)$ for *all* h, of class $C_{\cdot 1}(C_{1\cdot})$ if $T^{*n}h \to 0$ $(T^n h \to 0)$ for $h=0$ *only* and that $C_{\alpha\beta}=C_{\alpha\cdot} \cap C_{\cdot\beta}$ $(\alpha, \beta=0, 1)$. Note that in our case T_1, T_2 are of class $C_{\cdot 1}$, $C_{\cdot 0}$, respectively.

2. Main theorem. The main purpose of this paper is to prove the following

Theorem 1. *Let T be a completely non-unitary contraction on the separable Hilbert space $H(\neq \{0\})$ with the characteristic function $\Theta_T(\lambda)$. Let $\Theta_T(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$ be the canonical factorization of $\Theta_T(\lambda)$ into the product of its outer factor $\Theta_1(\lambda)$ and inner factor $\Theta_2(\lambda)$. Let $T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ be the triangulation of T corresponding to the decomposition $H=H_1 \oplus H_2$ induced by $\Theta_T(\lambda)=\Theta_2(\lambda)\Theta_1(\lambda)$. Then the following conditions are equivalent:*

(1) *T is similar to a direct sum of a unitary operator and a contraction of class $C_{\cdot 0}$;*

(2) *T_1 is similar to a unitary operator;*

(3) *$\Theta_1(\lambda)$ has a bounded analytic inverse.*

If this is the case, T is similar to $T_1 \oplus T_2$.

Proof.

(1)$\Rightarrow$(2):

Assume T is similar to $U \oplus V$ on the space $K=K_1 \oplus K_2$, where U is unitary on K_1 and V is a contraction of class $C_{\cdot 0}$ on K_2. Let S be an invertible operator from H onto K such that $T=S^{-1}(U \oplus V)S$. Consider $H_1'=S^{-1}K_1$ and $H_2'=H \ominus H_1'$. Obviously H_1' is an invariant subspace for T.

Let $T=\begin{bmatrix} T_1' & X' \\ 0 & T_2' \end{bmatrix}$ be the triangulation of T corresponding to the decomposition $H=H_1' \oplus H_2'$ and $S=\begin{bmatrix} S_1 & Y \\ 0 & S_2 \end{bmatrix}$ the triangulation corresponding to $H=H_1' \oplus H_2'$ and $K=K_1 \oplus K_2$. Note that S_2 is invertible since S and S_1 both are and the inverse of S is given by $S^{-1}=\begin{bmatrix} S_1^{-1} & -S_1^{-1}YS_1^{-1} \\ 0 & S_2^{-1} \end{bmatrix}$. We have

$$T=\begin{bmatrix} T_1' & X' \\ 0 & T_2' \end{bmatrix}=\begin{bmatrix} S_1^{-1} & -S_1^{-1}YS_2^{-1} \\ 0 & S_2^{-1} \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\begin{bmatrix} S_1 & Y \\ 0 & S_2 \end{bmatrix}=\begin{bmatrix} S_1^{-1}US_1 & * \\ 0 & S_2^{-1}VS_2 \end{bmatrix}.$$

It follows that $T_1'=S_1^{-1}US_1$ and $T_2'=S_2^{-1}VS_2$. T_1' and T_2' are of class $C_{\cdot 1}$, $C_{\cdot 0}$, respectively, since U and V are. It follows from the uniqueness of the triangulation of a contraction of type $\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}$ that H_1', H_2', T_1' and T_2' must coincide with H_1, H_2, T_1 and T_2, respectively (see [5], Sec. II. 4). Hence $T_1=S_1^{-1}US_1$ and $T_2=S_2^{-1}VS_2$. In particular, T_1 is similar to a unitary operator and T is similar to $T_1 \oplus T_2$.

(2)$\Leftrightarrow$(3):

Since the characteristic function of T_1 is the purely contractive part of $\Theta_1(\lambda)$, say $\Theta_1^0(\lambda)$, and $\Theta_1(\lambda)$ has a bounded analytic inverse if and only if $\Theta_1^0(\lambda)$ has, the equivalence of (2) and (3) follows from a theorem of Sz.-Nagy and Foiaş [5, Sec. IX. 1].

(3)$\Rightarrow$(1):

Assume $\Theta_1(\lambda)$ has a bounded analytic inverse $\Theta_1^{-1}(\lambda)$. We will work on the functional model of T.

Let S_1 be the operator from H_1 to $\overline{\Delta_1 L^2(\mathfrak{D}_T)}$ defined by

$$S_1(\Theta_2 u\oplus v) = v-\Delta_1\Theta_1^{-1}u \quad \text{for} \quad \Theta_2 u\oplus v\in H_1$$

and S the operator from H_2 to $\overline{\Delta_1 L^2(\mathfrak{D}_T)}$ defined by

$$S(u\oplus 0) = -\Delta_1\Theta_1^{-1}\Theta_2^* u \quad \text{for} \quad u\oplus 0\in H_2$$

where $\Delta_1(t)=[I-\Theta_1(e^{it})^*\Theta_1(e^{it})]^{1/2}$. Let U be multiplication by e^{it} on the space $\overline{\Delta_1 L^2(\mathfrak{D}_T)}$. Note that U is a unitary operator.

We want to show

(i) S_1 is invertible and $S_1^{-1}US_1=T_1$, (ii) $US-ST_2=S_1X$.

For the proof of (i), consider the space

$$H_1' = [H^2(\mathfrak{F})\oplus\overline{\Delta_1 L^2(\mathfrak{D}_T)}]\ominus\{\Theta_1 w\oplus\Delta_1 w\colon w\in H^2(\mathfrak{D}_T)\}$$

and the unitary operator W from H_1 to H_1' defined by

$$W(\Theta_2 u\oplus v) = u\oplus v \quad \text{for} \quad \Theta_2 u\oplus v\in H_1$$

(see [5], p. 290, proof of Prop. VII. 2. 1). Let $S_1'=S_1W^{-1}$, i.e. S_1' is the operator from H_1' to $\overline{\Delta_1 L^2(\mathfrak{D}_T)}$ given by $S_1'(u\oplus v)=v-\Delta_1\Theta_1^{-1}u$. Now it suffices to show S_1' is invertible. Let P be the restriction to H_1' of the orthogonal projection onto $\overline{\Delta_1 L^2(\mathfrak{D}_T)}$, i.e.

$$P(u\oplus v) = v \quad \text{for} \quad u\oplus v\in H_1'.$$

By our assumption, T_1 is similar to a unitary operator. It follows that P is an invertible operator and $P^*U=(WT_1W^{-1})P^*$ (see [5], p. 342, proof of Theorem IX. 1.2). We want to show $S_1'=(P^{-1})^*$. Equivalently,

$$(P^{-1}(v'), u\oplus v) = (v', S_1'(u\oplus v))$$

for any $u\oplus v\in H_1'$ and $v'\in\overline{\Delta_1 L^2(\mathfrak{D}_T)}$, where (,) denotes the corresponding inner product. Set $P^{-1}(v')=u'\oplus v'\in H_1'$. The last equation will be the same as

$$(u'\oplus v', u\oplus v) = (v', v-\Delta_1\Theta_1^{-1}u),$$

i.e.

$$(u', u)+(v', \Delta_1\Theta_1^{-1}u) = 0.$$

But $w=\Theta_1^{-1}u\in H^2(\mathfrak{D}_T)$ so that

$$(u', u)+(v', \Delta_1\Theta_1^{-1}u) = (u', \Theta_1 w)+(v', \Delta_1 w) = 0,$$

since $u'\oplus v'\in H_1'$.

Hence we proved $S_1'=(P^{-1})^*$ is invertible and satisfies $US_1'=S_1'WT_1W^{-1}$. Since $S_1'=S_1W^{-1}$, we have S_1 is invertible and $US_1W^{-1}=US_1'=S_1'WT_1W^{-1}= =S_1T_1W^{-1}$. Hence $US_1=S_1T_1$, or $S_1^{-1}US_1=T_1$ as asserted.

Now we verify (ii).

Consider any $u\oplus 0\in H_2$. Then $T(u\oplus 0)=(e^{it}u\oplus 0)-(\Theta_T w\oplus\Delta_T w)$ for some $w\in \in H^2(\mathfrak{D}_T)$ and $T_2(u\oplus 0)=T(u\oplus 0)-(\Theta_2 u'\oplus v')$ for some $\Theta_2 u'\oplus v'\in H_1$. Hence

$$\begin{aligned} T_2(u\oplus 0) &= (e^{it}u\oplus 0)-(\Theta_T w\oplus\Delta_T w)-(\Theta_2 u'\oplus v')= \\ &= (e^{it}u-\Theta_T w-\Theta_2 u')\oplus(-\Delta_T w-v'). \end{aligned}$$

Since $T_2(u\oplus 0)\in H_2$, we have $-\Delta_T w-v'=0$. Note that $X(u\oplus 0)=\Theta_2 u'\oplus v'$. Hence

$$\begin{aligned} (US-ST_2)(u\oplus 0) &= U(-\Delta_1\Theta_1^{-1}\Theta_2^* u)-S[(e^{it}u-\Theta_T w-\Theta_2 u')\oplus 0] = \\ &= e^{it}(-\Delta_1\Theta_1^{-1}\Theta_2^* u)-(-\Delta_1\Theta_1^{-1}\Theta_2^*)(e^{it}u-\Theta_T w-\Theta_2 u') = \\ &= -\Delta_1\Theta_1^{-1}\Theta_2^*\Theta_T w-\Delta_1\Theta_1^{-1}u' = -\Delta_1 w-\Delta_1\Theta_1^{-1}u' = \\ &= v'-\Delta_1\Theta_1^{-1}u' = S_1(\Theta_2 u'\oplus v') = S_1X(u\oplus 0). \end{aligned}$$

This proves (ii).

Hence

$$T=\begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}=\begin{bmatrix} S_1^{-1}US_1 & S_1^{-1}US-S_1^{-1}ST_2 \\ 0 & T_2 \end{bmatrix}=\begin{bmatrix} S_1^{-1} & -S_1^{-1}S \\ 0 & 1 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & T_2 \end{bmatrix}\begin{bmatrix} S_1 & S \\ 0 & 1 \end{bmatrix}=$$

$$=\begin{bmatrix} S_1 & S \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} U & 0 \\ 0 & T_2 \end{bmatrix}\begin{bmatrix} S_1 & S \\ 0 & 1 \end{bmatrix}.$$

This shows T is similar to $U\oplus T_2$ on the space $\overline{\Delta_1 L^2(\mathfrak{D}_T)}\oplus H_2$ and completes the proof.

For a geometric and simpler proof of more general facts than the implication (2)⇒(1) see [2]. However the above proof gives explicit forms for the operators which implement the similarities.

3. Some consequences. An immediate result of the preceding theorem is

Theorem 2. *Let T be as in Theorem 1. Then the following conditions are equivalent:*

(1) *T is similar to an isometry;*

(2) *T_1 is similar to a unitary operator and T_2 is similar to a unilateral shift;*

(3) *$\Theta_1(\lambda)$ has a bounded analytic inverse and $\Theta_2(\lambda)$ has a bounded analytic left-inverse.*

If this is the case, T is similar to $T_1\oplus T_2$.

Proof. Since T_2 is of class $C_{\cdot 0}$, it is similar to a unilateral shift if and only if $\Theta_2(\lambda)$ has a bounded analytic left-inverse (see [6], Theorem 2.4).

This gives characterizations of those c.n.u. contractions which are similar to isometries. Another one is given by Sz.-Nagy and Foiaş [6], which says T is similar to an isometry if and only if $\Theta_T(\lambda)$ has a bounded analytic left-inverse.

In order to prove the next theorem, we need the following

Lemma. *A c.n.u. normal contraction is of class C_{00}.*

Proof. Assume T is a c.n.u. normal contraction. Then $D_T = D_{T^*}$ and

$$\Theta_T(\lambda) = [-T+\lambda D_T(1-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T = (\lambda - T)(1-\lambda T^*)^{-1}|\mathfrak{D}_T,$$

which is obviously both inner and *-inner. Hence T is of class C_{00} (see [5], Sec. VI. 3).

Now we can give

Theorem 3. *Let T be as in Theorem 1. Then T is similar to a normal operator if and only if T_1 is similar to a unitary operator and T_2 is similar to a normal operator. If this is the case, T is similar to $T_1 \oplus T_2$.*

Proof. A normal contraction can be decomposed as the direct sum of a unitary operator and a c.n.u. normal contraction, the latter being of class C_{00} by the preceding lemma. The conclusion now follows from Theorem 1.

Recall that a contractive analytic function $\{\mathfrak{D}_1, \mathfrak{D}_2, \Theta(\lambda)\}$ is said to have the scalar multiple $\delta(\lambda)$, if $\delta(\lambda)$ is a scalar valued analytic function, $\delta(\lambda) \not\equiv 0$, and there exists a contractive analytic function $\{\mathfrak{D}_2, \mathfrak{D}_1, \Omega(\lambda)\}$ such that

$$\Omega(\lambda)\Theta(\lambda) = \delta(\lambda)I_{\mathfrak{D}_1}, \quad \Theta(\lambda)\Omega(\lambda) = \delta(\lambda)I_{\mathfrak{D}_2} \quad \text{for} \quad \lambda \in D.$$

A slightly different argument gives the following

Theorem 4. *Let T be as in Theorem 1. Assume, moreover, $\Theta_1(\lambda)$ admits a scalar multiple. Then T is similar to a hyponormal operator if and only if T_1 is similar to a unitary operator and T_2 is similar to a hyponormal operator. If this is the case, T is similar to $T_1 \oplus T_2$.*

Proof. We have only to prove the necessity part.

Assume T is similar to the hyponormal operator A on the space K. Let S be an invertible operator from H onto K such that $T = S^{-1}AS$. Consider $K_1 = SH_1$ and $K_2 = K \ominus K_1$. Let $A = \begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix}$ be the triangulation of A corresponding to the decomposition $K = K_1 \oplus K_2$. As before, we can show T_1 is similar to A_1 and T_2 is similar to A_2. Since T_1 is of class $C_{\cdot 1}$ whose characteristic function $\Theta_1^0(\lambda)$ admits a scalar multiple (cf. [5], Prop. V. 6.8), the spectrum $\sigma(T_1)$ is contained in the unit circle C (see [5], Prop. VI. 4.4). Hence the hyomormal operator A_1 has spectrum $\sigma(A_1)=\sigma(T_1)\subseteqq C$ of planar measure zero. It follows from a theorem of PUTNAM [3], that A_1 is indeed a unitary operator. Since A is hyponormal, this will imply that K_1 is a reducing subspace for A (see, e.g., [1], Sec. 0, Ex. 2). Therefore $X=0$ and A_2 is hyponormal. This completes the proof.

Note that in the preceding theorem, if we replace "hyponormal operator" by "subnormal operator", the corresponding conclusion is still true.

The theorems are stated for c.n.u. contractions, although they are still true for arbitrary contractions; however the proof of the general case along the above lines will involve some technical difficulties.

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$$\Phi(\lambda)\Theta_2(\lambda)+\Theta_1(\lambda)\Psi(\lambda) = I.$$

Added in proof. After the paper was submitted, C. R. PUTNAM (Hyponormal contractions and strong power convergence, *Pacific J. Math.*, **57** (1975), 531—538) showed that a c.n.u hyponormal contraction is of class $C_{\cdot 0}$. Hence it follows easily that Theorem 4 here holds even without assuming $\Theta_1(\lambda)$ admits a scalar multiple.

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Remark on the Jordan model for contractions of class $C_{\cdot 0}$

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In [5] SZ.-NAGY and C. FOIAŞ introduce a relation of complete injective-similarity, which is weaker than quasi-similarity [6; p. 70], and they use it to study operators of finite defect in class $C_{\cdot 0}$. In particular, they show that if Θ and Φ are quasi-equivalent $n \times m$ inner matrices over H^∞, then $S(\Theta)$ and $S(\Phi)$, the compressions of the unilateral shift of multiplicity m to the coinvariant subspaces determined by Θ and Φ, respectively, are completely injection-similar. This result partially extends, in a natural way, the theorem [1] which established, in the case $n=m<\infty$, that Θ and Φ are quasi-equivalent if and only if $S(\Theta)$ and $S(\Phi)$ are quasi-similar.

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Our object in this note is to show that in certain important cases the new relation of injection by a complete family can be replaced by the older and stronger relation of quasi-affine transform (see Theorem 2 and its Corollaries 1). One such case occurs when Φ is the normal form Θ' associated with Θ by the relation of quasi-equivalence; then on one side the result remains that there is a complete family of injections $\{X_1, X_2\}$ such that $S(\Theta)X_j = X_j S(\Theta')$, for $j=1, 2$; whereas, on the other side, our result will give the existence of a quasi-affinity X such that $XS(\Theta)=S(\Theta')X$.

Preliminaries

Let Θ and Φ be $n \times m$ matrices over the Hardy space H^∞ of bounded measurable functions on the unit circle T with vanishing Fourier coefficients of negative index. Such a matrix is called inner if $\Theta^*(e^{it})\Theta(e^{it})=I_m$ a.e. on T, where I_m is the $m \times m$ identity matrix. In this case it necessarily follows that $n \geqq m$. We will assume throughout that n is finite.

Associated with each inner Θ is a Hilbert space $\mathfrak{H}(\Theta)$ and an operator $S(\Theta)$ defined by

$$\mathfrak{H}(\Theta) = H_n^2 \ominus \Theta H_m^2 \quad \text{and} \quad S(\Theta)u = P_\theta(\chi u) \quad \text{for} \quad u \in \mathfrak{H}(\Theta),$$

where H_n^2 is the Hardy space of n dimensional (column) vector valued functions on $\boldsymbol{T}$, P_Θ is the orthogonal projection of H_n^2 onto $\mathfrak{H}(\Theta)$, and $\chi(z)=z$ for $z\in\boldsymbol{T}$. Operators of this type give canonical functional models for contractions in class $C_{\cdot 0}$ with finite defect. For a discussion of this operator class see [6].

A one-to-one operator X from a Hilbert space $\mathfrak{H}_1$ into a Hilbert space $\mathfrak{H}_2$ is called an *injection;* a family $\{X_\alpha\}$ of injections $X_\alpha\colon \mathfrak{H}_1\to\mathfrak{H}_2$ is called *complete* if the closed linear span of the ranges of the X_α's is $\mathfrak{H}_2$. If a complete family of injections consists of but a single operator, then the operator is called a *quasi-affinity.*

Suppose T_1 is an operator on $\mathfrak{H}_1$ and T_2 an operator on $\mathfrak{H}_2$. If there exists a complete injective family $\{X_\alpha\}$ such that $X_\alpha\colon \mathfrak{H}_2\to\mathfrak{H}_1$ and $T_1X_\alpha=X_\alpha T_2$, then T_2 is said to be *injected* into T_1 by $\{X_\alpha\}$, and we write $T_1\overset{ci}{\succ} T_2$. If $\{X_\alpha\}$ is a singleton, then T_2 is called a quasi-affine *transform* of T_1, and we write $T_1\succ T_2$. If $T_1\overset{ci}{\succ} T_2$ and $T_2\overset{ci}{\succ} T_1$ then T_1 and T_2 are said to be *completely injection-similar,* and we denote this by $T_1\overset{ci}{\sim} T_2$. This latter concept is an extension of *quasi-similarity* [6; p. 70], which can be viewed as the case when each family consists of a single quasi-affinity.

Finally we recall the definition of quasi-equivalence for $n\times m$ matrices over H^∞. Again let Θ and Φ be such matrices. Then Θ and Φ are said to be *quasi-equivalent* if for every inner function ω in H^∞ there exist an $n\times n$ matrix Δ and an $m\times m$ matrix Λ over H^∞ such that $\Delta\Theta=\Phi\Lambda$, and $\det\Delta\cdot\det\Lambda$ and ω are relatively prime. See [2] and [7].

A criterion for $S(\Theta)\prec S(\Phi)$

As mentioned in the introduction, Sz.-Nagy and C. Foiaş have shown [5; Theorem 1] that if Θ and Φ are quasi-equivalent $n\times m$ inner matrices over H^∞, then $S(\Theta)$ are completely injection-similar. Further, the two families of injections can always be chosen so as to consist of two operators each, and an example is given to show that a single injection may not suffice.

Before stating our main result, we note that the converse of their theorem is also true. Suppose $S(\Theta)$ and $S(\Phi)$ are completely injection-similar. If Θ' and Φ' are the quasi-equivalent normal forms[1] [2] of Θ and Φ, respectively, then $S(\Theta')\overset{ci}{\sim} S(\Theta)\overset{ci}{\sim} S(\Phi)\overset{ci}{\sim} S(\Phi')$. Further, it was shown in [5; Theorem 3] that $S(\Theta)$ is injection-

[1] The normal matrix corresponding to an $n\times m$ matrix Θ over H^∞ is the $n\times m$ matrix that has the j^{th} invariant factor θ_j of Θ in position jj for $1\leq j\leq m$ and zeros elsewhere. The invariant factor θ_j is the quotient δ_j/δ_{j-1} if $\delta_{j-1}\neq 0$, and 0 if $\delta_{j-1}=0$, where, $\delta_0=1$ and δ_j is the greatest common inner divisor of the j^{th} order minors of Θ.

similar to a unique Jordan operator; therefore, $S(\Theta')=S(\Phi')$, and hence $\Theta'=\Phi'$. Thus Θ and Φ are quasi-equivalent.

The following theorem gives our criterion.

Theorem 1. *Suppose θ and Φ are $n\times m$ matrices over H^∞. Necessary and sufficient conditions that $S(\theta)\prec S(\Phi)$ are that there exist square matrices Δ and Λ over H^∞ which statisfy*

(1) $\Delta\Theta = \Phi\Lambda$,

(2) $\ker[\Delta\ \Phi] \subseteqq \Theta H^2_m \oplus H^2_m$,

(3) $[\Delta\ \Phi]H^2_{n+m}$ *is dense in* H^2_n.

Remark. By $[\Delta\ \Phi]$ we mean the $n\times(n+m)$ matrix over H^∞ made up of the columns of Δ followed by those of Φ. By an abuse of notation (as in (2) above) we identify this matrix with the analytic Toeplitz operator from H^2_{n+m} to H^2_n that it induces. Similarly we will identify any matrix over H^∞ with an analytic Toeplitz operator when convenient.

Proof. Suppose there exists a quasi-affinity X from $\mathfrak{H}(\Theta)$ to $\mathfrak{H}(\Phi)$ such that

$$XS(\Theta) = S(\Phi)X.$$

By the lifting theorem (see [3] for the case of scalar $\Theta=\Phi$ and [6; p. 258] for the general case), there exists an $n\times n$ matrix Δ over H^∞ such that

$$X = P_\Phi\Delta|\mathfrak{H}(\Theta)$$

and $\Delta\Theta H^2_m \subseteqq \Phi H^2_n$. The latter condition is equivalent to the existence of a Λ satisfying (1).

Property (2) is most easily established by noting its equivalence to

(2') if $f\in H^2_n$ and $\Delta f\in H^2_m$, then $f\in\Theta H^2_m$.

To establish (2') suppose $f\in H^2_n$ and $\Delta f\in \Phi H^2_m$. Write $f=u+\Theta h$, where $u\in\mathfrak{H}(\Theta)$ and $h\in H^2_m$, and apply (1) to obtain

$$\Delta f = \Delta u + \Phi\Lambda h.$$

Since $\Delta f\in\Phi H^2_n$, an application of P_Φ yields

$$Xu = P_\Phi\Delta u = 0.$$

By the injectivity of X, $u=0$, and thus $f=\Theta h$, which establishes (2').

As for (3), the fact that X is a quasi-affinity implies $X\mathfrak{H}(\Theta)+\Phi H^2_m$ is dense in H^2_n. Since $Xf=P_\Phi\Delta f$, it follows that $X\mathfrak{H}(\Theta)+\Phi H^2_m$ is included in $[\Delta\ \Phi]H^2_{n+m}$. Therefore (3) holds.

Conversely, if there exists an $n\times n$ matrix Δ satisfying (1), (2), and (3), then define X to be $P_\Phi \Delta|\mathfrak{H}(\Theta)$. The argument that X is a quasi-affinity and satisfies $XS(\Theta)=S(\Phi)X$, is straightforward.

*

The following two lemmas essentially form the key ingredients in the proof of the injectivity part of Theorem 1 in [5]. We include them here for easy reference.

Lemma 1. *A sufficient condition for* (2) *to hold is the existence of Δ and Λ having determinants which are nonzero a.e. on* $\boldsymbol{T}$, *satisfying* (1), *and such that if $g\in L^2_m$, $\Lambda g\in H^2_m$ and $\Theta g\in H^2_n$, then $g\in H^2_m$.*

Proof. The question is: does $f\in H^2_n$ and $\Delta f\in \Phi H^2_m$ imply $f\in \Theta H^2_m$? Suppose h in H^2_m is such that $\Delta f=\Phi h$. Since the determinants of Δ and Λ are nonzero a.e., both Δ^{-1} and Λ^{-1} exist a.e. on T. Consequently, the following relations hold pointwise a.e. on $\boldsymbol{T}$:

$$\Delta f = \Phi\Lambda\Lambda^{-1}h = \Delta\Theta(\Lambda^{-1}h).$$

Thus

$$f = \Theta(\Lambda^{-1}h),$$

which implies $\Lambda^{-1}h\in L^2_m$, since $f\in H^2_m$ and Θ is isometric a.e. If $g=\Lambda^{-1}h$, then g satisfies the hypothesis, and hence $g\in H^2_m$. But $f=\Theta g$, and hence the answer to our question is yes.

Lemma 2. *A sufficient condition for* (2) *to hold is the existence of Δ and Λ satisfying* (1) *such that Δ has a determinant which is nonzero a.e. and Λ has a determinant relatively prime to the greatest common divisor of the $m\times m$ minors of Θ.*

Proof. By Lemma 1 it suffices to show that if $g\in L^2_m$, $\Lambda g\in H^2_m$ and $\Theta g\in H^2_n$, then $g\in H^2_m$. If the classical adjoint of Λ is applied to Λg, then we see that (det $\Lambda)g$ is in H^2_m. For any $m\times m$ submatrix Θ_α of Θ, we have $\Theta_\alpha g\in H^2_m$, and consequently (det $\Theta_\alpha)g\in H^2_m$. Since det Λ and the collection of all $m\times m$ minors of Θ form a relatively prime set, the conclusion follows from a lemma of Sz.-Nagy [4; p. 74].

*

On the basis of Lemma 2 we can obtain from Theorem 1:

Theorem 2. *Suppose Θ and Φ are quasi-equivalent $n\times m$ inner matrices over H^∞. If the rows of Φ span an m-dimensional subspace of H^2_m, then $S(\Theta)\prec S(\Phi)$.*

Proof. Select Δ_1 and Λ satisfying (1) such that each of their determinants is relatively prime to all the invariant factors of Θ and Φ.

By hypothesis, elementary row operations with complex scalars can be used to replace the last $n-m$ rows of Φ by rows of zeros, i.e. there exists an invertible $n\times n$ matrix A over $\mathbf{C}$ such that $A\Phi$ has the form $\begin{bmatrix}\Phi_1\\0\end{bmatrix}$ where Φ_1 is an $m\times m$ matrix over H^∞, and 0 is the $(n-m)\times m$ zero matrix. Let Δ_0 be the $(n-m)\times n$ matrix formed by the last $n-m$ rows of $A\Delta_1$. The closure $\mathfrak{M}$ of $\Delta_0 H_n^2$ is a full invariant subspace of the unilateral shift in H^2_{n-m}. (It is full since $\det\Delta_1\neq 0$ implies that at least one $(n-m)\times(n-m)$ minor of Δ_0, say δ, is nonzero. Hence $\mathfrak{M}$ includes δH^2_{n-m}.) Thus there exists an inner $(n-m)\times(n-m)$ matrix Ψ such that $\mathfrak{M}=\Psi H^2_{n-m}$. Set

$$\Delta = A^{-1}(I_m\oplus\Psi^*)A\Delta_1.$$

Then Δ is analytic since $\Psi^*\Delta_0$ is analytic, and from $(I_m\oplus\Psi^*)A\Phi=A\Phi$ we obtain

$$\Delta\Theta = A^{-1}(I_m\oplus\Psi^*)A\Delta_1\Theta = A^{-1}(I_m\oplus\Psi^*)A\Phi\Lambda = A^{-1}A\Phi\Lambda = \Phi\Lambda.$$

Thus Δ and Λ satisfy (1). From the definition of Ψ and Λ, we see that $\det\Delta$ divides $\det\Delta_1$, and thus $\det\Delta$ is relatively prime to the invariant factors of Θ and Φ.

Condition (2) now follows from Lemma 2. We shall show that $[\Delta\ \Phi]$ satisfies (3) by showing that if $\mathfrak{N}=[A\Delta\ A\Phi]H^2_{n+m}$, then $\mathfrak{N}$ is dense in H^2_n; this is equivalent because of the invertibility of A. It is convenient to regard H^2_n as the direct sum $H^2_m\oplus H^2_{n-m}$. Note that $A\Delta H^2_n$ includes $(\det\Delta)H^2_n$, which in turn includes $(\det\Delta)H^2_m\oplus\{0\}$, and also $A\Phi H^2_m$ includes $(\det\Phi_1)H^2_m\oplus\{0\}$. Hence $\mathfrak{N}$ includes the sum of the two manifolds $(\det\Delta)H^2_m\oplus\{0\}$ and $(\det\Phi_1)H^2_m\oplus\{0\}$. But $\det\Delta$ and $\det\Phi_1$ are relatively prime, and thus Beurling's theorem implies that $\overline{\mathfrak{N}}$ includes $H^2_m\oplus\{0\}$. From the fact that $\mathfrak{N}$ includes $A\Delta H^2_n$, it now follows that $\overline{\mathfrak{N}}$ also includes $\{0\}\oplus\Psi^*\Delta_0 H^2_n$, and hence $\overline{\mathfrak{N}}\supset\{0\}\oplus\Psi^*\mathfrak{M}=\{0\}\oplus H^2_{n-m}$. Thus $\overline{\mathfrak{N}}=H^2_n$.

Corollary 1. *If Θ is $n\times m$ inner and Θ' is its normal form, then $S(\Theta)\prec S(\Theta')$.*[2)]

Proof. Immediate from Theorem 2.

Finally, for any operator T on a Hilbert space $\mathfrak{H}$ the multiplicity μ_T is defined to be the minimal cardinality of a set $\mathfrak{M}$ in $\mathfrak{H}$ such that

$$\mathfrak{H} = \bigvee_{j=0}^{\infty} T^j\mathfrak{M}.$$

In [5; Proposition 3] it is shown, in particular, that if Θ' is the normal form of Θ, then

$$\mu_{S(\Theta)} \leqq 2\mu_{S(\Theta')}.$$

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2) In the special case that Θ is also *-outer (and hence $S(\Theta)\in C_{10}$) this result is contained in [5], Corollary 2.

This follows from a general observation that if $T_1 \overset{ci}{\succ} T_2$ and if $X=\{X_\alpha\}$ is a corresponding complete system of injections, then $\mu_{T_1} \leqq (\text{card}(X)) \cdot \mu_{T_2}$.

By Corollary 1 we can add the following to Proposition 3 of [5].

Corollary 2. *If Θ is $n\times m$ inner over H^∞ and Θ' is its quasi-equivalent normal form, then*

$$\mu_{S(\Theta')} \leqq \mu_{S(\Theta)} \leqq 2\mu_{S(\Theta')}.$$

Proof. Proposition 3 of [5] and Corollary 1.

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Compléments à l'étude des opérateurs de classe C_0. III

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I. Modèle de Jordan

1. Dans cette partie de la Note on étend des résultats des articles [1], [2] sur le modèle de Jordan des opérateurs de classe C_0, de multiplicité finie, aux opérateurs de multiplicité quelconque, définis dans un espace de Hilbert séparable $\mathfrak{H}$. Voir aussi [H].

Faisant usage de notations déjà employées dans les articles cités, considérons les opérateurs

$$S(M) = S(m_1) \oplus S(m_2) \oplus \ldots \tag{1.1}$$

attachés à des suites $M = \{m_j\}_1^\infty$ de fonctions intérieures m_j, telles que m_{j+1} soit un diviseur de m_j pour $j = 1, 2, \ldots$. Nous convenons de ne pas distinguer deux fonctions interieures qui coïncident à un facteur numérique constant près. Notons que l'espace $\mathfrak{H}(m)$, dans lequel $S(m)$ est défini, se réduit à l'espace zéro $\{0\}$ si $m = 1$ et dans ce cas seulement. La multiplicité μ de $S(M)$ est égale à ∞ $(= \aleph_0)$ si toutes les fonctions m_j sont non-constantes, autrement elle est égale au rang j de la dernière des fonctions m_j non-constantes.

Nous allons établir le suivant

Théorème 1. *Pour un opérateur T de classe C_0, danc un espace de Hilbert séparable $\mathfrak{H}$, il existe un opérateur $S(M)$ de type* (1.1) *et un seul tel que*

$$T \succ S(M). \tag{1.2}$$

Cet opérateur $S(M)$ vérifie aussi la relation $S(M) \succ T$, donc est quasi-similaire à T.

2. Pour un opérateur T de classe C_0 dans l'espace $\mathfrak{H}$ on désignera par m_T la fonction minimum de T et par m_f $(f \in H)$ la fonction minimum de la restriction $T_f = T|\mathfrak{H}_f$ où

où $$\mathfrak{H}_f = \bigvee_{n=0}^{\infty} T^n f.$$

Dans la démonstration du théorème on fera usage en particulier des propositions suivantes:

Proposition 1. *Pour T de classe C_0 dans $\mathfrak{H}$ les vecteurs $f \in \mathfrak{H}$ tels que*

$$m_f = m_T, \tag{2.1}$$

sont denses dans $\mathfrak{H}$.

Démonstration. L'existence d'un $g \in \mathfrak{H}$ vérifiant $m_g = m_T$ est assurée par le théorème 1 de [2]. Soit h un élément donné quelconque de $\mathfrak{H}$ et soit $[g, h]$ le sous-espace (de dimension $\leqq 2$) sous-tendu par g et h. D'après le lemme de SHERMAN (cf. [2], Lemme 3) les vecteurs f de $[g, h]$ pour lesquels $m_f = m_g \vee m_h$ $(= m_T \vee m_h = m_T)$ sont denses dans $[g, h]$. Par conséquent, les f vérifiant (2.1) sont denses dans $\mathfrak{H}$.

Proposition 2. *Soit f un élément de l'espace $\mathfrak{H}$ vérifiant* (2.1). *Il existe alors un sous-espace $\mathfrak{M}$ de $\mathfrak{H}$ invariant pour T et une quasi-affinité*

$$X: \mathfrak{H}(m) \to \mathfrak{H}_f \qquad (m = m_T) \tag{2.2}$$

tels que

$$XS(m) = TX, \tag{2.3}$$

$$\mathfrak{H}_f \vee \mathfrak{M} = \mathfrak{H}, \tag{2.4}$$

$$X\mathfrak{H}(m) \cap \mathfrak{M} = \{0\}. \tag{2.5}$$

Démonstration. Il n'y a qu'à combiner la démonstration donnée dans la section 2 de [1] avec celle de la Proposition 2 de [2], p. 295.

3. Cela étant nous abordons la démonstration du théorème en montrant qu'il existe un opérateur $S(M)$ vérifiant (1.2).

Choisissons, à cet effet, une suite $\{\varphi_m\}_1^\infty$ de vecteurs dans $\mathfrak{H}$ qui sous-tendent $\mathfrak{H}$, et soit $\{\psi_n\}_1^\infty$ une suite dans laquelle chaque φ_m se repète à une infinité de fois.

En vertu de la Proposition 1 il existe un $f_1 \in \mathfrak{H}$ tel que

$$m_{f_1} = m_T \quad \text{et} \quad \|f_1 - \psi_1\| < \frac{1}{2}. \tag*{(3.1)$_1$}$$

D'après la Proposition 2 il existe alors un sous-espace $\mathfrak{M}_1$ de $\mathfrak{H}$ invariant pour T, et une quasi-affinité

$$X_1: \mathfrak{H}(m_1) \to \mathfrak{H}_{f_1} \qquad (m_1 = m_T) \tag*{(3.2)$_1$}$$

vérifiant

$$X_1 S(m_1) = TX_1, \tag*{(3.3)$_1$}$$

$$\mathfrak{H}_{f_1} \vee \mathfrak{M}_1 = \mathfrak{H}, \tag*{(3.4)$_1$}$$

$$X\mathfrak{H}(m_1) \cap \mathfrak{M}_1 = \{0\}. \tag*{(3.5)$_1$}$$

Vu $(3,1)_1$ on a aussi, puisque $f_1 \in \mathfrak{H}_{f_1}$,

$$(3.6)_1 \qquad \|\psi_1 - P_{\mathfrak{H}_{f_1}}\psi_1\| < \frac{1}{2}.$$

D'autre part, de $(3.4)_1$ nous déduisons qu'il existe un $f_{12} \in \mathfrak{H}_{f_1}$ et un $h_2 \in \mathfrak{M}_1$ tels que

$$(3.7) \qquad \|f_{12} + h_2 - \psi_2\| < \frac{1}{2^3}.$$

Appliquons maintenant les Propositions 1 et 2 à $T_1 = T|\mathfrak{M}_1$ au lieu de T et notons que m_{T_1}, que nous désignons aussi par m_2, est un diviseur de $m_1 (= m_T)$. Nous obtenons qu'il existe un $f_2 \in \mathfrak{M}_1$ tel que

$$(3.1)_2 \qquad m_{f_2} = m_2, \qquad \|f_2 - h_2\| < \frac{1}{2^3},$$

un sous-espace $\mathfrak{M}_2$ de $\mathfrak{M}_1$, invariant pour T_1 (donc pour T), et une quasi-affinité

$$(3.2)_2 \qquad X_2\colon \mathfrak{H}(m_2) \to \mathfrak{H}_{f_2},$$

telles que

$$(3.3)_2 \qquad X_2 S(m_2) = TX_2,$$

$$(3.4)_2 \qquad \mathfrak{H}_{f_2} \vee \mathfrak{M}_2 = \mathfrak{M}_1,$$

$$(3.5)_2 \qquad X_2 \mathfrak{H}(m_2) \cap \mathfrak{M}_2 = \{0\}.$$

En vertu de (3.7) et $(3.1)_2$ on a aussi

$$\|f_{12} + f_2 - \psi_2\| < \frac{1}{2^3} + \frac{1}{2^3} = \frac{1}{2^2};$$

puisque $f_{12} \in \mathfrak{H}_{f_1}$ et $f_2 \in \mathfrak{H}_{f_2}$ cela entraîne

$$(3.6)_2 \qquad \|\psi_2 - P_{\mathfrak{H}_{f_1} \vee \mathfrak{H}_{f_2}}\psi_2\| < \frac{1}{2^2}.$$

On continue le procédé par récurrence et on obtient ainsi une suite

$$(\mathfrak{H} =)\, \mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \ldots \supset \mathfrak{M}_n \supset \ldots$$

de sous-espaces invariants pour T, des vecteurs

$$f_n \in \mathfrak{M}_{n-1} \qquad (n = 1, 2, \ldots)$$

tels que

$$(3.1)_n \qquad m_{f_n} = m_n \qquad (\text{où } m_n = m_{T|\mathfrak{M}_{n-1}}),$$

et des quasi-affinités

$$(3.2)_n \qquad X_n\colon \mathfrak{H}(m_n) \to \mathfrak{H}_{f_n} \qquad (n = 1, 2, \ldots)$$

telles que

$$(3.3)_n \qquad X_n S(m_n) = TX_n,$$

et que de plus on a

$$(3.4)_n \qquad \mathfrak{H}_{f_n} \vee \mathfrak{M}_n = \mathfrak{M}_{n-1},$$

$$(3.5)_n \qquad X_n \mathfrak{H}(m_n) \cap \mathfrak{M}_n = \{0\}$$

et

$$(3.6)_n \qquad \|\psi_n - P_{\mathfrak{H}_{f_1} \vee \ldots \vee \mathfrak{H}_{f_n}} \psi_n\| < \frac{1}{2^n} \qquad (n = 2, 3, \ldots).$$

On peut évidemment supposer aussi que

$$(3.8) \qquad \sum_n \|X_n\|^2 \leqq 1,$$

en exigeant p. ex. que $\|X_n\| < 1/2^n$.

Puisque $\mathfrak{M}_n \subset \mathfrak{M}_{n-1}$, m_n est un diviseur de m_{n-1}, donc la suite

$$M = \{m_1, m_2, \ldots, m_n, \ldots\}$$

est de type considéré dans la section 1. Il peut arriver que $\mathfrak{M}_n$ se réduise à $\{0\}$ à partir d'un certain rang n; pour tels n on a $m_n = 1$ et $\mathfrak{H}(m_n) = \{0\}$. Mais en tous les cas on peut former l'espace

$$(3.9) \qquad \mathfrak{H}(M) = \bigoplus_1^\infty \mathfrak{H}(m_n)$$

et l'opérateur $S(M)$ correspondant. De plus, grâce à (3.2), (3.3) et (3.8) on peut définir par

$$X(h_1 \oplus h_2 \oplus \ldots \oplus h_n \oplus \ldots) = \sum_1^\infty X_n h_n$$

un opérateur

$$X \colon \mathfrak{H}(M) \to \mathfrak{H}$$

tel que

$$\|X\| \leqq 1 \quad \text{et} \quad XS(M) = TX.$$

Nous allons démontrer que X est une quasi-affinité.

Soit $h = h_1 \oplus h_2 \oplus \ldots$ tel que $Xh = 0$, il s'agit de montrer que $h = \{0\}$. En effet, si $h \neq 0$, il existe un premier j tel que $h_j \neq 0$, et par (3.2) et (3.4) on a alors

$$X_j \mathfrak{H}(m_j) \ni X_j h_j = -\sum_{j+1}^\infty X_n h_n \in \bigvee_{j+1}^\infty \mathfrak{H}_{f_n} \subset \bigvee_{j+1}^\infty \mathfrak{M}_{n-1} \doteq \mathfrak{M}_j;$$

vu (3.5) cela entraîne $X_j h_j = 0$: contradiction parce que X_j a le noyau $\{0\}$. Donc X a aussi le noyau $\{0\}$.

D'autre part, la fermeture de $X\mathfrak{H}(M)$ étant évidemment égale à $\mathfrak{H}'=\bigvee_1^\infty \mathfrak{H}_{f_n}$, il nous reste à démontrer que $\mathfrak{H}'=\mathfrak{H}$.

Notons, à cet effet, que par (3.6) on a

$$\|\psi_n - P_{\mathfrak{H}'}\psi_n\| < \frac{1}{2^n} \quad \text{pour} \quad n = 1, 2, \ldots. \tag{3.10}$$

Or pour chaque élément φ_m du système $\{\varphi_m\}_1^\infty$ il existe une suite d'indices $n_j\to\infty$ telle que $\psi_{n_j}=\varphi_m$, donc par (3.10),

$$\|\varphi_m - P_{\mathfrak{H}'}\varphi_m\| < \frac{1}{2^{n_j}} \to 0 \quad (j\to\infty),$$

et par conséquent $\varphi_m - P_{\mathfrak{H}'}\varphi_m=0$, $\varphi_m\in\mathfrak{H}'$. Comme les φ sous-tendent $\mathfrak{H}$, cela prouve que $\mathfrak{H}'=\mathfrak{H}$.

Donc X est une quasi-affinité et on a

$$T \succ S(M).$$

4. Comme T^* est de classe C_0 en même temps que T, il existe aussi une suite $M'=\{m_j\}_1^\infty$ de même type que M pour laquelle $T^*\succ S(M')$, donc $T\prec S(P)$ où $P=\{p_j\}_1^\infty$, $p_j=m_j'^{\sim}$, et par conséquent

$$S(P)\succ T\succ S(M). \tag{4.1}$$

Soient $M=\{m_j\}$ et $P=\{p_j\}$ quelconques, satisfaisant

$$S(P)\succ S(M) \tag{4.2}$$

On a alors $p_j=m_j$ $(j=1, 2, \ldots)$. Cela se démontre par la méthode employée dans [1] et [3], pp. 313—316, notamment de la manière suivante.

Pour une fonction intérieure quelconque w nous formons les suites

$$P^w = \{p_j^w\} \quad \text{et} \quad M^w = \{m_j^w\} \qquad (j=1, 2, \ldots),$$

où $p_j^w=p_j/(p_j\wedge w)$ et $m_j^w=m_j/(m_j\wedge w)$; suites qui sont de même type que nous avons toujours envisagé ci-dessus, et de (4.2) il résulte aussi

$$S(P^w)\succ S(M^w). \tag{4.3}$$

Une conséquence immédiate de (4.3) est que, pour un entier k quelconque, $k\geqq 1$, l'opérateur

$$S(M_1^w) = S(m_k^w)\oplus\ldots\oplus S(m_k^w)$$

peut être injecté dans $S(P^w)$ dans le sens introduit dans [3], c'est-à-dire qu'il existe un opérateur injectif Y_k tel que $S(P^w)\,Y_k=Y_k\,S(M_k^w)$.

Choisissons $w=p_k$. Dans ce cas $p_j^w=1$ pour $j\geqq k$, donc $S(P^w)$ se réduit à

$$S(P_{k-1}^w) = S(p_1^w)\oplus\dots\oplus S(p_{k-1}^w).$$

Du fait que $S(M_k^w)$ peut être injecté dans $S(P_{k-1}^w)$ il s'ensuit, en vertu du théorème 4 de [3], que $m_k^w=1$, c'est-à-dire que $m_k\wedge w=m_k$, m_k est un diviseur de $w=p_k$.

Comme (4.2) entraîne $S(M)^*\succ S(P)^*$, donc

(4.2)* $$S(M^\sim)\succ S(P^\sim),$$

il s'ensuit de la même manière que $p_k^\sim$ est un diviseur de $m_k^\sim$, ce qui est équivalent à ce que p_k est un diviseur de m_k.

On conclut que $p_k=m_k$ $(k=1, 2, \dots)$, ce qui achève la démonstration de l'unicité de $S(M)$ vérifiant (1.2), et par (4.1) aussi de ce que T est quasi-similaire à $S(M)$.

Cela achève la démonstration du théorème 1.

II. Le bicommutant

5. Dans la Note [1] on a démontré (théorème 3) que le bicommutant $(T)''$ d'un opérateur T de classe C_0, de multiplicité finie, est constitué des opérateurs X fonctions de T, $X=\varphi(T)$, où φ appartient à la classe N_T de fonctions analytiques dans le disque unité, considérée dans le chap. IV de [H].

Faisant usage de notre théorème 1 sur le modèle jordanien d'un opérateur T de classe C_0, de type général, ainsi que d'autres résultats, on va étendre ce théorème à tous ces opérateurs T. On établira donc:

Théorème 2. *Pour un opérateur quelconque T de classe C_0, dans un espace de Hilbert séparable $\mathfrak{H}$, tout opérateur $X\in(T)''$ est de la forme*

$$X = \varphi(T) \quad \textit{où} \quad \varphi = u/v\in N_T.$$

6. Démonstration. Soit $S(M)=S(m_1)\oplus S(m_2)\oplus\dots$ le modèle de T, défini dans l'espace

(6.1) $$\mathfrak{H}(M) = \mathfrak{H}(m_1)\oplus\mathfrak{H}(m_2)\oplus\dots,$$

soient A, B des quasi-affinités telles que

(6.2) $$S(M)A = AT, \quad TB = BS(M),$$

et soit $X\in(T)''$. On a alors pour $Y\in\big(S(M)\big)'$ quelconque,

$$BYA\cdot T = BY\cdot AT = BY\cdot S(M)A = B\cdot YS(M)\cdot A = B\cdot S(M)Y\cdot A =$$
$$= BS(M)\cdot YA = TB\cdot YA = T\cdot BYA$$

et par conséquent BYA permute à X, d'où

$$AXB\cdot Y\cdot AB = AB\cdot Y\cdot AXB.$$

Notons aussi que (6.2) entraîne $AB\in(S(M))'$.

On aussi $AXB\in(S(M))'$ parce que

$$S(M)A\cdot XB = AT\cdot XB = A\cdot TX\cdot B = A\cdot XT\cdot B = AX\cdot TB = AX\cdot B\ S(M).$$

Donc en posant

$$R = AB \quad \text{et} \quad Q = AXB \tag{6.3}$$

on a R, $Q\in(S(M))'$ et l'équation

$$RYQ = QYR \tag{6.4}$$

est vérifiée pour tout $Y\in(S(M))'$.

Or $S(M)$ a pour dilatation isométrique minimum l'opérateur $S_\infty=S\oplus S\oplus\ldots$ dans l'espace $H^2_\infty=H^2\oplus H^2\oplus\ldots$ où S est la translation unilatérale $u(\lambda)\to\lambda\cdot u(\lambda)$ dans H^2. Donc, en représentant tout opérateur Φ dans $\mathfrak{H}(M)$ par sa matrice $[\Phi_{ij}]$ $(i, j=1, 2, \ldots)$ suivant la décomposition (6.2), on déduit du théorème sur la dilatation des commutants (cf. [H], Theorem VI. 3.6) que la forme générale d'un opérateur $Y=[Y_{ij}]$ dans $\mathfrak{H}(M)$, permutant à $S(M)$, est donnée par

$$Y_{ij}h_j = P_{\mathfrak{H}(m_i)}y_{ij}h_j \qquad (h_j\in\mathfrak{H}(m_j)) \tag{6.5}$$

où

$$y_{ij}\in H^\infty, \quad y_{ij}m_j\in m_iH^2, \tag{6.6}$$

$$\sum_{i=1}^{\infty}\left|\sum_{j=1}^{\infty}y_{ij}(\lambda)c_j\right|^2 \leqq \|Y\|^2\cdot\sum_{j=1}^{\infty}|c_j|^2 \tag{6.7}$$

pour tout $c=(c_1, c_2, \ldots)\in l^2$ et tout λ complexe, $|\lambda|<1$.

Soient $[r_{ij}]$ et $[q_{ij}]$ les matrices sur H^∞ correspondant dans ce sens à R et Q, et choisissons pour k fixé quelconque $(k=1, 2, \ldots)$ l'opérateur $Y^{(k)}$ pour lequel la matrice $[y_{ij}^{(k)}]$ est donnée par

$$y_{ij}^{(k)} = \begin{cases} 1 & \text{si} \quad (i, j) = (k, 1) \\ 0 & \text{dans tous les autres cas;} \end{cases} \tag{6.8}$$

les relations (6.6) et (6.7) sont évidemment vérifiées, avec $\|Y^{(k)}\|=1$, puisque m_k est un diviseur de m_1.

En appliquant à (6.4) la propriété de multiplication donnée dans [4], p. 227, on aura

$$P_{\mathfrak{H}(m_i)}(r_{ik}q_{1j}-q_{ik}r_{1j})h = 0 \quad \text{pour} \quad h\in\mathfrak{H}(m_j).$$

La même équation subsiste pour $h \in \mathfrak{H}(m_j)^\perp$ $(=m_j H^2)$ parce que de (6.6) il s'ensuit

$$r_{ik} q_{1j} m_j \in r_{ik} m_1 H^2 = r_{ik} m_k \cdot \frac{m_1}{m_k} H^2 \subset m_i H^2$$

et la relation analogue pour q, r au lieu de r, q.

On conclut que

$$P_{\mathfrak{H}(m_i)}(q_{1j} r_{ik} - r_{1j} q_{ik}) h = 0 \quad \text{pour} \quad i, j, k = 1, 2, \ldots \quad \text{et pour} \quad h \in H^2. \tag{6.9}$$

Prenons en particulier $h \in \mathfrak{H}(m_k)$. En utilisant les relations (6.5) et (6.6) on déduit de (6.9) que

$$q_{1j}(S(m_i)) R_{ik} - r_{1j}(S(m_i)) Q_{ik} = 0 \quad \text{pour} \quad i, j, k = 1, 2, \ldots,$$

d'où il résulte

$$q_{1j}(S(M)) R = r_{1j}(S(M)) Q \quad \text{pour} \quad j = 1, 2, \ldots. \tag{6.10}$$

Comme les relations (6.2) s'étendent aux fonctions de classe H^∞ des opérateurs $S(M)$ et T, il dérive de (6.10) et de (6.3) que

$$A q_{1j}(T) B = A r_{1j}(T) X B.$$

Puisque A et B sont des quasi-affinités il en résulte que

$$q_{1j}(T) = r_{1j}(T) X \quad \text{pour} \quad j = 1, 2, \ldots. \tag{6.11}$$

En vertu de (6.7) on a en particulier

$$\left| \sum_{j=1}^{\infty} r_{1j}(\lambda) c_j \right| \leqq \|R\| \|c\| \quad \text{pour tout} \quad c \in l^2 \quad \text{et} \quad |\lambda| < 1; \tag{6.12}$$

par conséquent la série au premier membre converge uniformément pour $|\lambda| < 1$ vers une somme appartenant à H^2, et même à H^∞, et la correspondance

$$\tau \colon c \to \sum_{j=1}^{\infty} r_{1j}(\lambda) c_j$$

définit un opérateur

$$\tau \colon l^2 \to H^\infty (\subset H^2)$$

tel que $\|\tau\| \leqq \|R\|$. On a:

$$\bigvee_{n \geqq 0} (S(m_1))^n P_{\mathfrak{H}(m_1)} \tau l^2 = \bigvee_{n \geqq 0,\, j \geqq 1} P_{\mathfrak{H}(m_1)} \lambda^n r_{1j} \mathbf{C} = \bigvee_j P_{\mathfrak{H}(m_1)} r_{1j} H^2 = \bigvee_j P_{\mathfrak{H}(m_1)} r_{1j} \mathfrak{H}(m_j)$$

parce que $r_{1j} m_j \in m_1 H^2$,

$$= \bigvee_j R_{1j} \mathfrak{H}(m_j) = \overline{P_1 R \mathfrak{H}(M)}$$ où P_1 est la projection de $\mathfrak{H}(M)$ à son premiei espace composant dans la somme (6.1),

$$= P_1 \mathfrak{H}(M)$$ parce que $R = AB$ est une quasi-affinité,

$$= \mathfrak{H}(m_1).$$

Cela veut dire que le sous-ensemble linéaire $\tau_1 l^2$ de $\mathfrak{H}(m_1)$, où

$$\tau_1 = P_{\mathfrak{H}(m_1)}\tau,$$

est cyclique pour $S(m_1)$, c'est-à-dire que

$$\bigvee_{n\geqq 0} (S(m_1))^n(\tau_1 l^2) = \mathfrak{H}(m_1).$$

Nous appliquons maintenant le lemme qu'on formulera après. Il s'ensuit qu'il existe un $c\in l^2$ tel que $f=P_{\mathfrak{H}(m_1)}\tau c$ vérifie la relation

$$m_f = m_{S(m_1)} \quad (=m_1), \tag{6.13}$$

m_f étant la fonction minimum de la restriction de $S(m_1)$ au sous-espace invariant engendré par f. Or il est facile à montrer que pour tout $g\in\mathfrak{H}(m_1)$, $g\neq 0$ on a

$$m_g = \bigwedge_{w\in A(m_1, g)} w$$

où $A(m_1, g)$ est l'ensemble des fonctions intérieures w pour lesquelles wg est divisible par m_1. Il s'ensuit, en vertu de [1], p. 106, que

$$m_g = \frac{m_1}{m_1\wedge g}.$$

La relation (6.13) est ainsi équivalente à ce que $m_1/(m_1\wedge f)=m_1$, ou $m_1\wedge f=1$, c'est-à-dire que m_1 et f n'ont pas de diviseur intérieur commun non-constant.

Or il est évident que $m_1\wedge f=m_1\wedge\tau c$. Donc il existe $c\in l^2$ tel que

$$v = \sum_j c_j r_{1j}\in H^\infty, \quad m_1\wedge v = 1.$$

Considérons aussi la série $\sum_j c_j q_{1j}$. Par des raisons analogues, notamment par (6.7), cette série converge aussi uniformément dans $|\lambda|<1$ vers une fonction $u\in H^\infty$. De (6.11) il dérive

$$u(T) = v(T)X.$$

Puisque $v\wedge m_1=1$, $v(T)$ est une quasi-affinité, *cf.* [H], Propositions III. 3.3 et III. 4.7 b.

Par conséquent on a

$$X = \varphi(T) \quad \text{où} \quad \varphi = u/v\in N_T.$$

7. Cela achève la démonstration du théorème 2 sauf qu'on a encore à formuler et établir le lemme dont on s'est servi dans la démonstration.

Lemme. *Soit $T \in C_0$ dans un espace de Hilbert $\mathfrak{H}$, et soit $\sigma: \mathfrak{K} \to \mathfrak{H}$ un opérateur* (linéaire, continu) *d'un espace $\mathfrak{K}$ de Banach dans $\mathfrak{H}$ tel que*

$$\mathfrak{H} = \bigvee_{n \geq 0} T^n(\sigma \mathfrak{K}).$$

Il existe alors un $k \in \mathfrak{K}$ tel que $m_{\sigma k} = m_T$.

La démonstration est essentiellement la même que celle du théorème 1 dans [2], sauf qu'il y a à appliquer le théorème de catégorie de Baire à $\mathfrak{K}$ au lieu de $\mathfrak{H}$.

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Untersuchungen über äquivalente Variations-probleme von mehreren Veränderlichen

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§ 1. Einleitung

Es sei $F(x^i, \dot{x}^i_\alpha)$ $(i=1, 2, \ldots, n; \alpha=1, 2, \ldots, m; 1 \leqq m < n)$ eine Funktion der $n(m+1)$ Veränderlichen x^i und $\dot{x}^i_\alpha$, ferner nehmen wir an, daß $F(x^i, \dot{x}^i_\alpha)$ in den x^i mindestens einmal, in den $\dot{x}^i_\alpha$ aber mindestens dreimal stetig differenzierbar ist. Mit Hilfe dieser Funktion sei ein m-parametriges Integral von der Form:

$$\mathscr{I}(F) \stackrel{\text{def}}{=} \int_{\mathscr{A}_m} \ldots \int F\left(x^i(u^1, u^2, \ldots, u^m) \frac{\partial x^i}{\partial u^\alpha}\right) du^1 \ldots du^m \tag{1.1}$$

festgelegt, wo $\mathscr{A}_m$ ein m-dimensionales Bereich des n-dimensionalen Punktraumes X_n bedeutet. Der Euler—Lagrangesche Operator des Integrals (1.1) ist:

$$\mathscr{E}_i(F) \stackrel{\text{def}}{=} \frac{\partial F}{\partial x^i} - \frac{d}{du^\alpha} \frac{\partial F}{\partial \dot{x}^i_\alpha}, \quad \dot{x}^i_\alpha \stackrel{\text{def}}{=} \frac{\partial x^i}{\partial u^\alpha}, \tag{1.2}$$

wo jetzt, und im folgenden die Einsteinsche Summationskonvention auf doppelt vorkommende Indizes immer gelten soll. Die $\dot{x}^i_\alpha$ bilden m Tangentenvektoren des Unterraumes $x^i(u^\alpha)$; diese sollen immer linear-unabhängig sein.

Ist nun (1.1) das Grundintegral eines Variationsproblems, so genügen die extremalen Unterräume bekanntlich dem Differentialgleichungssystem

$$\mathscr{E}_i(F) = 0. \tag{1.3}$$

Zwei Variationsprobleme mit den Grundfunktionen $F(x^i, \dot{x}^i_\alpha)$ und $F^*(x^i, \dot{x}^i_\alpha)$ nennen wir äquivalent, falls die Schar ihrer extremalen Unterräume übereinstimmt.

In den Arbeiten [1] und [2] haben wir gewisse Type von äquivalenten Variationsproblemen untersucht. In [1] war das Grundintegral ein einparametriges parameterinvariantes Integral; in [2] war aber das Grundintegral ein $(n-1)$-parametriges Integral, die Parameterinvarianz war aber bei diesem Fall nicht bedingt. Im folgenden seien nun m-parametrige parameterinvariante Integrale von der Form (1.1) zu Grunde gelegt. Die Parameterinvarianz bedeutet, daß die Funktion $F(x^i, \dot{x}^i_\alpha)$ den Relationen

$$(1.4)\qquad (\partial_{\dot{x}^i_\alpha} F)\dot{x}^i_\beta = \delta^\alpha_\beta F(x, \dot{x}_\gamma), \qquad (x\colon x^1, \dots, x^n;\ \dot{x}_\gamma\colon \dot{x}^1_\gamma, \dots, \dot{x}^n_\gamma)$$

genügt (vgl. [3], S. 268). Jetzt und im folgenden werden die lateinischen Indizes immer die Zahlen $1, 2, \dots, n$; die griechischen aber die Zahlen $1, 2, \dots, m$ durchlaufen. Die in (1.4) verwandte Bezeichnungen $(x, \dot{x}_\gamma)$ bedeuten: $(x^i, \dot{x}^i_\gamma)$ (vgl. (1.2)). Aus (1.4) erhält man nach einer partiellen Ableitung nach $\dot{x}^k_\varrho$ die im späteren wichtige Relation (vgl. [3], S. 270):

$$(1.5)\qquad (\partial^2_{\dot{x}^i_\alpha \dot{x}^k_\varrho} F)\dot{x}^i_\beta \equiv \delta^\alpha_\beta \partial_{\dot{x}^k_\varrho} F - \delta^\varrho_\beta \partial_{\dot{x}^k_\alpha} F.$$

Wir wollen jetzt solche äquivalente Variationsprobleme untersuchen, deren Euler—Lagrangesche Operatoren miteinander durch die Formeln

$$(1.6)\qquad \mathscr{E}_i(F^*) \equiv \lambda^k_i(x)\mathscr{E}_k(F), \qquad \lambda^k_i(x) \neq 0,$$

verbunden sind, wo die Funktionen $\lambda^k_i(x)$ einen nur vom Orte x^i abhängigen gemischten Tensor bedeuten, und die Relation (1.6) längs jedes Unterraumes

$$x^i = x^i(u^1, u^2, \dots, u^m)$$

bestehen soll. Da der Euler—Lagrangescher Operator vektoriellen Charakter hat, ist (1.6) eine koordinateninvariante Tensorrelation.

In diesem Aufsatz wollen wir nun beweisen, daß unter gewissen Bedingungen die Formel (1.6) im wesentlichen nur dann möglich ist, falls

$$(1.7)\qquad \lambda^k_i(x) = \delta^k_i \varphi(x)$$

besteht, wo $\varphi(x)$ einen allein vom Orte x^i abhängigen Skalar bezeichnet. Wäre statt des Typs (1.6) die Relation

$$(1.8)\qquad \mathscr{E}_i(F^*) \equiv \varphi(x, \dot{x}_\varrho)\mathscr{E}_i(F), \qquad \varphi(x, \dot{x}_\varrho) \neq 0$$

längs jedes Unterraumes $x^i(u^\alpha)$ gültig, wo $\varphi(x, \dot{x}_\varrho)$ einen Skalar bedeutet, so würde (vgl. Hauptsatz II.):

$$(1.9)\qquad (\partial_{\dot{x}^k_\varrho} \varphi(x, \dot{x}_\alpha))\dot{x}^k_\sigma = 0$$

bestehen. Die Grundfunktionen in den Formeln (1.6) und (1.8) bestimmen offenbar äquivalente Variationsprobleme, da aus (1.3) evident auch $\mathscr{E}_i(F^*)=0$ folgt. Das ist auch umgekehrt richtig, falls $\operatorname{Det}(\lambda^k_i)\neq 0$ ist, denn in diesem Fall existiert der inverse Tensor von λ^k_i.

§ 2. Der Typ $\mathscr{E}_i(F^*) \equiv \lambda_i^k(x)\mathscr{E}_k(F)$

Es seien

$$F(x^i(u^\alpha),\ \dot{x}^i_\gamma(u^\beta)), \quad F^*(x^i(u^\alpha),\ \dot{x}^i_\gamma(u^\beta))$$

zwei Funktionen, die Grundintegrale von der Form (1.1) bestimmen. Nehmen wir ferner an, daß die Relation (1.6) längs jedes Unterraumes $x^i(u^\alpha)$ besteht. Die Relation (1.6) kann auf Grund von (1.2) in der Form

$$\partial_{x^i}F^* - \lambda_i^k\,\partial_{x^k}F - \dot{x}^j_\alpha(\partial^2_{x^j\dot{x}^i_\alpha}F^* - \lambda_i^k\,\partial^2_{x^j\dot{x}^k_\alpha}F) - \ddot{x}^j_{\alpha\beta}(\partial^2_{\dot{x}^i_\alpha\dot{x}^j_\beta}F^* - \lambda_i^k\,\partial^2_{\dot{x}^k_\alpha\dot{x}^j_\beta}F) \equiv 0, \tag{2.1}$$

$$\ddot{x}^j_{\alpha\beta} \overset{\text{def}}{=} \frac{\partial^2 x^j}{\partial u^\alpha\,\partial u^\beta}$$

angegeben werden, woraus wegen $\ddot{x}^j_{\alpha\beta} \equiv \ddot{x}^j_{\beta\alpha}$ unmittelbar folgt, daß

$$\partial^2_{\dot{x}^i_{(\alpha}\dot{x}^j_{\beta)}}F^* \equiv \lambda_i^k(x)\,\partial^2_{\dot{x}^k_{(\alpha}\dot{x}^j_{\beta)}}F \text{ [1]} \tag{2.2}$$

besteht. $\ddot{x}^j_{\alpha\beta}$ kommt nämlich in den einzelnen Gliedern von (2.1) nur im letzten vor und wegen der Identität muß sein Koeffizient verschwinden. Die linke Seite von (2.2) ist aber auch in (i,j) symmetrisch, wie das durch das explizite Aufschreiben der linken Seite unmittelbar verifiziert werden kann. Somit muß aber auch die rechte Seite in (i,j) symmetrisch sein, d. h. der schiefsymmetrische Teil der rechten Seite von (2.2) ist Null. Es ist somit:

$$\lambda_i^k(x)\,\partial^2_{\dot{x}^k_{(\alpha}\dot{x}^j_{\beta)}}F - \lambda_j^k(x)\,\partial^2_{\dot{x}^k_{(\alpha}\dot{x}^i_{\beta)}}F \equiv 0. \tag{2.3}$$

Eine Kontraktion mit $\dot{x}^i_\varrho$ gibt nun auf Grund von (1.5)

$$\lambda_i^k(x)\dot{x}^i_\varrho\,\partial^2_{\dot{x}^k_{(\alpha}\dot{x}^j_{\beta)}}F \equiv 0. \tag{2.4}$$

Bemerkung. Da

$$\dot{x}^i_\varrho\mathscr{E}_i(F) \equiv 0 \tag{2.5}$$

immer eine Identität in x^i, $\dot{x}^i_\alpha$, $\ddot{x}^i_{\alpha\beta}$ ist (vgl. [3], Kap 4. (5.30)), muß der Koeffizient von $\ddot{x}^i_{\alpha\beta}$ verschwinden; das gibt aber eben die bei der Herleitung von (2.4) benützte Relation

$$\dot{x}^i_\varrho\,\partial^2_{\dot{x}^k_{(\alpha}\dot{x}^i_{\beta)}}F \equiv 0, \tag{2.6}$$

was aber selbstverständlich auch mit (1.5) leicht bewiesen werden kann.

Wir beschränken uns im folgenden auf solche Type der Variationsprobleme, deren Grundfunktionen der folgenden Forderung genügen.

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[1]) Die Klammern bei den Indizes bedeuten in (2.2) den in (α, β) symmetrischen Teil.

Forderung. *Es soll die Grundfunktion $F(x, \dot{x}_\gamma)$ des Grundintegrals* (1.1) *bei fest gewählten α und β der Relation*

$$Rang\,(\partial^2_{\dot{x}^k_{(\alpha}\dot{x}^j_{\beta)}}F) = n-m$$

genügen, wo der Klammerausdruck eine quadratische Matrix n-ter Ordnung bedeutet.

Auf Grund der Forderung hat das Gleichungssystem

$$X^k_\sigma\,\partial^2_{\dot{x}^k_{(\alpha}\dot{x}^j_{\beta)}}F = 0 \qquad (\alpha, \beta\text{: fest}) \tag{2.7}$$

bezüglich X^k_σ genau m linear-unabhängige Lösungen: $X^k_1, X^k_2, \ldots, X^k_m$. Nach der Formel (2.6) sind das eben die Tangentenvektoren $\dot{x}^k_\varrho$ des Unterraumes $x^k(u^\alpha)$, sogar für jede Indizes α, β in (2.6) bilden sie dasselbe linear-unabhängige Lösungssystem. Nach (2.4) muß aber dann

$$\lambda^k_i(x)\dot{x}^i_\varrho = \Phi^\alpha_\varrho(x, \dot{x}_\gamma)\dot{x}^k_\alpha \tag{2.8}$$

bestehen, wo die Φ^α_ϱ von $(x, \dot{x}_\gamma)$ abhängige Skalare bezeichnen.

Differenzieren wir (2.8) partiell nach $\dot{x}^j_\sigma$, so wird:

$$\lambda^k_j(x)\,\delta^\sigma_\varrho = \delta^k_j\Phi^\sigma_\varrho(x, \dot{x}_\gamma) + (\partial_{\dot{x}^j_\sigma}\Phi^\alpha_\varrho)\dot{x}^k_\alpha. \tag{2.9}$$

Eine Verjüngung bezüglich ϱ, σ gibt unmittelbar:

$$\lambda^k_j(x) = \delta^k_j\varphi(x, \dot{x}_\gamma) + m^{-1}(\partial_{\dot{x}^j_\beta}\Phi^\alpha_\beta)\dot{x}^k_\alpha \tag{2.10}$$

mit

$$\varphi(x, \dot{x}_\gamma) \overset{\text{def}}{=} m^{-1}\Phi^\alpha_\alpha(x, \dot{x}_\gamma), \tag{2.10a}$$

wo $\varphi(x, \dot{x}_\gamma)$ offenbar eine skalare Funktion bedeutet. Setzen wir nun von der Formel (2.10) $\lambda^k_j(x)$ in die zu Grunde gelegte Formel (1.6) ein, beachten wir ferner die Identität (2.5), so erhalten wir:

$$\mathscr{E}_i(F^*) \equiv \varphi(x, \dot{x}_\varrho)\mathscr{E}_i(F), \qquad \varphi(x, \dot{x}_\gamma) \neq 0, \tag{2.11}$$

was formal mit (1.8) übereinstimmt; die Funktion φ ist aber durch (2.10a) festgelegt.

Wenn wir die aus (1.6) abgeleitete Formel: (2.11) mit der ursprünglichen Formel (1.6) vergleichen und dann beachten, daß in (1.6) die Koeffizienten $\lambda^k_i(x)$ von $\mathscr{E}_k(F)$ von den Größen $\dot{x}^l_\gamma$ unabhängig sind, so folgt, daß die Type (1.6) und (2.11) dann und nur dann übereinstimmen, falls der Skalar $\varphi(x, \dot{x}_\gamma)$ im wesentlichen von den $\dot{x}^l_\gamma$ unabhängig ist, d. h. (1.7) gilt. Der Ausdruck: „im wesentlichen" deutet auf (2.10), wo in der Formel von λ^k_j die Größen $\dot{x}^l_\gamma$ doch vorhanden sind, aber sie müssen — wie das schon bemerkt wurde — aus der Formel (2.11) herausfallen, falls (2.11) die Form (1.6) hat. Somit folgt der

Hauptsatz I. *In m-parametrigen äquivalenten Variationsproblemen mit parameterinvarianten Grundintegralen und für die* (1.6) *besteht, hat $\lambda^k_i(x)$ im wesentlichen —*

d. h. in seiner ursprünglichen, in (1.6) *vorkommenden Form — die Gestalt* (1.7), *d. h.* $\varphi(x)$ *ist von den* $\dot{x}^i_\gamma$ *unabhängig, bzw.* $\lambda^k_i(x)$ *kann in der Form* (2.10), (2.10a) *angegeben werden.*

Bemerkung. $\varphi(x)$ in (1.7) braucht im allgemeinen nicht mit $\varphi(x, \dot{x}_\gamma)$ von (2.11) übereinstimmen. Aus (2.11) müssen aber die Größen $\dot{x}^k_\alpha$ von $\varphi(x, \dot{x}_\gamma)$ herausfallen.

Bezüglich der Größen Φ^α_β wollen wir noch auf Grund von (2.8) bzw. (2.10) einige Relationen ableiten. Die Bedeutung der Funktionen Φ^α_β besteht darin, daß auf Grund von (2.5) und (2.6) die $\Phi^\alpha_\varrho \dot{x}^k_\alpha$ die allgemeinen gemeinsamen Lösung aller Gleichungen von der Form (2.7) bzw.

$$X^k_\sigma \mathscr{E}_k(F) = 0$$

sind. Wir wollen aber betonen, daß in diesem Paragraphen in unseren nachfolgenden Untersuchungen die Gültigkeit der Formel (2.8) immer vorausgesetzt wird. Das bedeutet die Existenz einer solchen Funktion $F^*(x, x_\gamma)$, die mit $F(x, x_\gamma)$ durch (1.6) verbunden ist. Wir beweisen den

Satz 1. $\Phi^\alpha_\beta(x, \dot{x}_\gamma)$ *ist in den* $\dot{x}^i_\gamma$ *homogen von nullter Dimension, falls die* $\dot{x}^i_\gamma$ *linear unabhängig sind.*

Beweis. Eine Überschiebung von (2.9) mit $\dot{x}^j_\sigma$ gibt in Hinsicht auf (2.8):

$$(\partial_{\dot{x}^j_\sigma} \Phi^\alpha_\varrho)\dot{x}^j_\sigma \dot{x}^k_\alpha = 0. \tag{2.12}$$

Es ist aber das Gleichungssystem

$$\dot{x}^k_\alpha y^\beta_k = \delta^\beta_\alpha \tag{2.13}$$

bezüglich der y^β_k immer lösbar, falls der Rang der Matrix $(\dot{x}^k_\alpha)$ gleich m ist. Das ist aber wegen der linearen Unabhängigkeit der $\dot{x}^k_\gamma$ erfüllt; somit erhält man nach einer Überschiebung mit y^β_k von (2.12)

$$(\partial_{\dot{x}^j_\sigma} \Phi^\beta_\varrho)\dot{x}^j_\sigma = 0, \tag{2.14}$$

was nach Euler eben die charakteristische Differentialgleichung der in den $\dot{x}^k_\alpha$ von nullter Dimension homogenen Funktionen ist. Damit ist der Satz 1. bewiesen.

Wir werden jetzt eine Relation für die in (2.10a) definierte Funktion $\varphi(x, \dot{x}_\gamma)$ ableiten.

Satz 2. *Es gilt für* $\varphi(x, \dot{x}_\gamma)$ *die Relation*

$$(\partial_{\dot{x}^j_\sigma} \varphi)\dot{x}^j_\beta = 0, \quad \varphi \overset{\text{def}}{=} m^{-1}\Phi^\alpha_\alpha, \tag{2.15}$$

falls (2.8) *besteht und die inversen Größen* y^β_k *von* $\dot{x}^k_\alpha$ *existieren.*

Bemerkung. Dieser Satz ist im wesentlichen von den Variationsproblemen unabhängig.

Beweis. Aus (2.8) folgen die Relationen (2.9) und (2.10). Eine Überschiebung von (2.9) mit y_k^ϱ gibt auf Grund von (2.13) und (2.10a)

$$\lambda_j^k y_k^\sigma = \Phi_\varrho^\sigma y_j^\varrho + m\,\partial_{\dot{x}_\sigma^j}\varphi.$$

Eine neue Überschiebung mit $\dot{x}_\beta^j$ gibt in Hinsicht auf (2.8) und (2.13) eben die Relation (2.15), w. z. b. w.

Im falle $m=1$ sind offensichtlich (2.14) und (2.15) identisch und drücken die Homogenität nullter Dimension von $\varphi(x, \dot{x})$ in den $\dot{x}^i$ aus.

§ 3. Der Typ $\mathscr{E}_i(F^*) \equiv \varphi(x, \dot{x}_\varrho)\mathscr{E}_i(F)$

In diesem Paragraphen wollen wir den Typ (1.8) untersuchen, falls aber (1.6), (2.10) und (2.10a) nicht bestehen. Auf Grund von (1.2) hat (1.8) die Form:

$$\partial_{x^i}F^* - \varphi\,\partial_{x^i}F - \dot{x}_\alpha^j(\partial^2_{x^j\dot{x}_\alpha^i}F^* - \varphi\,\partial^2_{x^j\dot{x}_\alpha^i}F) - \ddot{x}_{\alpha\beta}^j(\partial^2_{\dot{x}^i_{(\alpha}\dot{x}^j_{\beta)}}F^* - \varphi\,\partial^2_{\dot{x}^i_{(\alpha}\dot{x}^j_{\beta)}}F) \equiv 0. \tag{3.1}$$

Wegen der Identität in allen $x^i(u^\varrho)$, $\dot{x}_\alpha^i(u^\varrho)$ verschwindet der Koeffizient von $\ddot{x}_{\alpha\beta}^j$. Schreiben wir das auf, differenzieren wir noch nach $\dot{x}_\gamma^k$, so wird:

$$\partial^3_{\dot{x}^i_{(\alpha}\dot{x}^j_{\beta)}\dot{x}^k_\gamma}F^* \equiv \varphi(x, \dot{x}_\delta)\,\partial^3_{\dot{x}^i_{(\alpha}\dot{x}^j_{\beta)}\dot{x}^k_\gamma}F + (\partial_{\dot{x}^k_\gamma}\varphi)\,\partial^2_{\dot{x}^i_{(\alpha}\dot{x}^j_{\beta)}}F. \tag{3.2}$$

Differenzieren wir nun (1.5) nach $\dot{x}_\sigma^j$, so wird:

$$(\partial^3_{\dot{x}^i_\alpha\dot{x}^j_\sigma\dot{x}^k_\varrho}F)\dot{x}_\beta^i \equiv \delta_\beta^\alpha\,\partial^2_{\dot{x}^k_\varrho\dot{x}^j_\sigma}F - \delta_\beta^\varrho\,\partial^2_{\dot{x}^k_\alpha\dot{x}^j_\sigma}F - \delta_\beta^\sigma\,\partial^2_{\dot{x}^j_\alpha\dot{x}^k_\varrho}F.$$

Überschieben wir jetzt (3.2) mit $\dot{x}_\sigma^k$, beachten wir dann unzere letzte Identität und ferner, daß

$$\partial^2_{\dot{x}^i_{(\alpha}\dot{x}^j_{\beta)}}F^* - \varphi(x, \dot{x}_\sigma)\,\partial^2_{\dot{x}^i_{(\alpha}\dot{x}^j_{\beta)}}F \equiv 0, \tag{3.3}$$

was immer besteht, da die linke Seite eben der Koeffizient von $\ddot{x}_{\alpha\beta}^j$ in (3.1) ist, so bekommt man:

$$(\partial_{\dot{x}^k_\gamma}\varphi)\dot{x}_\sigma^k\,\partial^2_{\dot{x}^i_{(\alpha}\dot{x}^j_{\beta)}}F \equiv 0. \tag{3.4}$$

Aus dieser Formel bekommt man den

Satz 3. *Aus der Identität* (1.8) *folgt, daß entweder* (1.9), *oder*

$$\partial^2_{\dot{x}^i_{(\alpha}\dot{x}^j_{\beta)}}F \equiv 0 \tag{3.5}$$

gültig ist.

Nehmen wir jetzt an, daß (3.5) gültig ist. Die Formel (3.1) reduziert sich wegen (3.3) auf

(3.6) $$\partial_{x^i} F^* - \varphi\, \partial_{x^i} F - \dot{x}^j_\alpha (\partial^2_{x^j \dot{x}^i_\alpha} F^* - \varphi\, \partial^2_{x^j \dot{x}^i_\alpha} F) \equiv 0.$$

Differenzieren wir diese Identität partiell nach $\dot{x}^k_\beta$, überschieben wir dann mit $\dot{x}^k_\varrho$, beachten wir die Relationen (1.4) und (1.5), die wir noch nach x^j partiell ableiten müssen, so erhält man in Hinsicht auf (3.6) selbst:

(3.7) $$(\partial_{\dot{x}^k_\beta} \varphi)\, \dot{x}^k_\varrho (\partial_{x^i} F - \dot{x}^j_\alpha\, \partial^2_{x^j \dot{x}^i_\alpha} F) \equiv 0.$$

Aus dieser Formel folgt der

Satz 4. *Sind* (1.8) *und* (3.5) *Identitäten längs jedes Unterraumes* $x^i(u^\alpha)$, *so besteht entweder* (1.9) *oder es ist* $\mathscr{E}_i(F) \equiv 0$.

Beweis. Wäre (1.9) nicht gültig, so würde aus (3.5) und aus dem Verschwinden des Faktors von $(\partial_{\dot{x}^k_\beta} \varphi)\, \dot{x}^k_\varrho$ in (3.7) unmittelbar $\mathscr{E}_i(F) \equiv 0$ folgen, wie beauptet wurde.

Aus den Sätzen 3 und 4 folgt der

Haupsatz II. *Ist* (1.8) *gültig, ferner* $\mathscr{E}_i(F) \not\equiv 0$, *so besteht für* $\varphi(x, \dot{x}_\gamma)$ *die Relation* (1.9).

Beweis. Wäre (1.9) nicht richtig, so müßte nach dem Satz 3 die Relation (3.5) bestehen. Somit würde aber nach Satz 4 $\mathscr{E}_i(F) \equiv 0$, im Widerspruch zu unserer Annahme.

Schließlich wollen wir noch denjenigen Fall untersuchen, in dem statt (3.3) die stärkere Bedingung

(3.8) $$\partial^2_{\dot{x}^i_\alpha \dot{x}^j_\beta} F^* - \varphi(x, \dot{x}_\gamma)\, \partial^2_{\dot{x}^i_\alpha \dot{x}^j_\beta} F \equiv 0$$

besteht. Eine Überschiebung dieser Identität mit $\dot{x}^j_\varrho$ gibt auf Grund von (1.5)

$$\delta^\beta_\varrho\, \partial_{\dot{x}^i_\alpha} F^* - \delta^\alpha_\varrho\, \partial_{\dot{x}^i_\beta} F^* - \varphi(x, \dot{x}_\gamma)(\delta^\beta_\varrho\, \partial_{\dot{x}^i_\alpha} F - \delta^\alpha_\varrho\, \partial_{\dot{x}^i_\beta} F) \equiv 0.$$

Nehmen wir an, daß $m > 1$ ist, so gibt eine Verjüngung über ϱ und β die Identität:

(3.9) $$\partial_{\dot{x}^i_\alpha} F^* - \varphi(x, \dot{x}_\gamma)\, \partial_{\dot{x}^i_\alpha} F \equiv 0.$$

Nach einer neuen Überschiebung mit $\dot{x}^i_\alpha$ wird im Hinblick auf (1.4):

(3.10) $$F^*(x, \dot{x}_\gamma) - \varphi(x, \dot{x}_\delta)\, F(x, \dot{x}_\gamma) \equiv 0.$$

Differenzieren wir nun diese Identität partiell nach $\dot{x}^i_\alpha$, subtrahieren wir dann aus (3.9), so wird:

$$(\partial_{\dot{x}^i_\alpha} \varphi)\, F(x, \dot{x}_\gamma) \equiv 0,$$

woraus folgt:

Satz 5. *Ist* $m>1$, *so folgt aus der Identität* (3.8), *daß die Funktion* $\varphi(x, \dot{x}_\gamma)$ *in* (3.10) *von den* $\dot{x}^i_\gamma$ *unabhängig ist.*

Auf Grund von (3.10) und nach Satz 5 geht (1.8) in

$$\mathscr{E}_i(\varphi(x)F) \equiv \varphi(x)\mathscr{E}_i(F), \qquad F = F(x, \dot{x}_\gamma)$$

über. Nach (1.2) bedeutet diese Relation:

$$\partial_{x^i}\varphi - \frac{d\varphi}{du^\alpha}\partial_{\dot{x}^i_\alpha}F \equiv 0,$$

oder, nach einer kleinen Umformung:

$$(\partial_{x^k}\varphi)(\delta^k_i - \dot{x}^k_\alpha\partial_{\dot{x}^i_\alpha}F) \equiv 0$$

woraus nach Satz 5 der folgende dritte Haupsatz folgt:

Hauptsatz III. *Ist* $m>1$, *ferner ist*

$$Det\,(\delta^k_i - \dot{x}^k_\alpha\partial_{\dot{x}^i_\alpha}F) \neq 0,$$

so folgt aus (1.8) *und* (3.8), *daß in* (1.8) *die Funktion* $\varphi(x, \dot{x}_\varrho)$ *eine Konstante ist.*

Zum Schluß bemerken wir noch, daß wenn φ eine Konstante ist, so bestimmen

$$\mathscr{E}_i(F) = 0, \quad \mathscr{E}_i(F^*) \equiv \varphi\mathscr{E}_i(F) = 0$$

wirklich äquivalente Variationsprobleme. Die Grundfunktionen F und F^* können aber selbstverständlich verschieden sein, wie das das Korollar auf Seite 111 von [1] zeigt. In diesem Beispiel war aber $m=1$. Ist $m=n-1$, so verweisen wir auf den Satz 3 von [2] auf S. 278. Nach diesem Satz folgt aus unserer Bedingung (3.8), falls $n>2$ und $m=n-1$ gelten, und

$$\mathscr{E}_i(F^*(x, \dot{x}_\gamma)) \equiv \mathscr{E}_i(F(x, \dot{x}_\gamma)),$$

daß im parameterinvarianten Fall $F^*(x, \dot{x}_\gamma) \equiv F(x, \dot{x}_\gamma)$ ist. Dieser Satz zeigt also, daß in manchen Fällen die Parameterinvarianz im Fall $m>1$ eine stärkere Bedingung ist als im Fall $m=1$.

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On the theory of Finsler connections especially their equivalence

By P. T. NAGY in Szeged

§ 1. Introduction

It is well known that the solution of the equivalence problem of linear connections can be formulated with help of the torsion and curvature tensors of the connection (cf. [1] pp. 74—78, [4] p. 77.).

In the case of Varga's theory of line-element connections the conditions of equivalence is formulated with help of a set of tensors differing from the torsion and curvature tensors defined by the structure equations [3]. The Matsumoto's Finsler connection theory [2] can be regarded as a generalization of Varga's theory and hence it seems to be important to discuss the equivalence theory in this case too.

The purpose of our paper is to deal with the equivalence problem using Varga's method.

The terminology of Matsumoto's monograph [2] will be used throughout.

§ 2. Preliminaries

Let $T(M)$ be the tangent bundle of a differentiable manifold M. A non-linear connection N defined by a distribution $y \in TM \to N_y$ in $T(TM)$ satisfying $T_y TM = N_y \oplus T_y^v$, namely the tangent space $T_y TM$ and the vertical subspace T_y^v. Let $L(M)$ be the linear frame bundle of M. The Finsler bundle is defined by $\pi_T^{-1} L(M)$, where π_T is the projection map of $T(TM)$,

A non-linear connection N determines the connection map $K: TTM \to TM$ so that for any $z \in T_y TM$ the vertical lift $l^v K(z)$ of the vector $K(z) \in T_{\pi_T(y)} M$ is the vertical component of z. The kernel of this map K is the horizontal subspace N_y of $T_y TM$.

A Finsler connection is a pair (Γ, N) of a connection Γ in the Finsler bundle $F(M)$ (called the *directional connection*) and a non-linear connection N in the tangent bundle $T(M)$.

There are three characteristic linear forms and a vector field on the Finsler bundle, namely the *horizontal basic form* or *h-basic form*

$$\theta_u^{(h)} = u^{-1} \circ d\pi_T \circ d\pi_F,$$

which is independent from the Finsler connection (π_F is the projection in the Finsler bundle $F(M)$),

the *vertical basic form* or *v-basic form*

$$\theta_u^{(v)} = u^{-1} \circ K \circ d\pi_F,$$

which is determined by the non-linear connection N,

the *connection form* ω, which is determined by the directional connection Γ in $F(M)$, and

the *supporting element* $\varepsilon_u = u^{-1} \circ \pi_F$. (Here $u \in F(M)$ is considered as a linear mapping of R^n onto $T_x M$, where $x = \pi_T \circ \pi_F(u)$.)

The values of $\theta^{(h)}$, $\theta^{(v)}$ and ε are in the vector space R^n and the values of the connection form ω are in the Lie algebra $\mathfrak{gl}(n)$.

Let be U a coordinate neighborhood in M with a local coordinate system $x^1, \dots, x^n$. Let $y^1, \dots, y^n$ and $z_1^1, \dots, z_n^1, z_1^2, \dots, z_{n-1}^n, z_n^n$ be the coordinates of the tangent vectors and the liner frames, with respect to the frame $\dfrac{\partial}{\partial x^1}, \dots, \dfrac{\partial}{\partial x^n}$. So we have $(x^1, \dots, x^n; y^1, \dots, y^n; z_1^1, \dots, z_n^n)$ as local coordinates in $\pi_F^{-1} \circ \pi_T^{-1}(U) \subset F(M)$. In terms of this coordinate system the forms $\theta^{(h)}$, $\theta^{(v)}$, ω and the vector field ε can be expressed as follows:

$$\theta^{(h)} = \theta^{(h)a} e_a = (z^{-1})_i^a \, dx^i e_a, \tag{1}$$

$$\theta^{(v)} = \theta^{(v)a} e_a = (z^{-1})_i^a (dy^i + N_j^i \, dx^j) e_a = (z^{-1})_i^a \delta y^i e_a, \tag{2}$$

$$\omega = \omega_b^a E_a^b = (z^{-1})_i^a (dz_b^i + z_b^j F_j{}^i{}_k \, dx^k + z_b^j C_j{}^i{}_k \delta y^k) E_a^b, \tag{3}$$

$$\varepsilon = \varepsilon^a e_a = (z^{-1})_i^a y^i e_a. \tag{4}$$

Here $e_1, \dots, e_n$ is a basis for R^n, $E_1^1, \dots, E_n^n$ is a basis for $\mathfrak{gl}(n)$, $(z^{-1})_i^a$ are the elements of the matrix inverse to z_b^i and we used the notation

$$\delta y^i = dy^i + N_j^i \, dx^j.$$

§ 3. Characteristic tensors

The invariants (torsions and curvatures) of the Finsler connection are defined by the structure equations (cf. in an equivalent form [2], 18 §.)

$$d\theta^{(h)a} = -\omega^a_b \wedge \theta^{(h)b} + (z^{-1})^a_i \left(\tfrac{1}{2} T_i{}^l{}_k \, dx^k \wedge dx^i + C_i{}^l{}_k \delta y^k \wedge dx^i\right), \tag{5}$$

$$d\theta^{(v)a} = -\omega^b_a \wedge \theta^{(v)b} + (z^{-1})^a_i \left(P_i{}^l{}_k \, dx^k \wedge \delta y^i + \tfrac{1}{2} S_i{}^l{}_k \delta y^k \wedge \delta y^i + \tfrac{1}{2} R_i{}^l{}_k \, dx^k \wedge dx^i\right), \tag{6}$$

$$d\omega^a_b = -\omega^a_c \wedge \omega^c_b + (z^{-})^a_i z^j_b \left(\tfrac{1}{2} R_j{}^i{}_{mk} \, dx^k \wedge dx^m + P_j{}^i{}_{mk} \, dx^k \wedge \delta y^m + \tfrac{1}{2} S_j{}^i{}_{mk} \delta y^k \wedge \delta y^m\right) \tag{7}$$

as follows:

h-torsion tensors:

$$T_i{}^l{}_k = F_i{}^l{}_k - F_k{}^l{}_i, \quad C_i{}^l{}_k; \tag{8a}$$

v-torsion tensors:

$$\begin{aligned} R_i{}^l{}_k &= \frac{\partial N^l_i}{\partial x^k} - \frac{\partial N^l_i}{\partial y^m} N^m_k - \frac{\partial N^l_k}{\partial x^i} + \frac{\partial N^l_k}{\partial y^m} N^m_i, \\ P_i{}^l{}_k &= F_i{}^l{}_k - \frac{\partial N^l_k}{\partial y^i}, \\ S_i{}^l{}_k &= C_i{}^l{}_k - C_k{}^l{}_i; \end{aligned} \tag{8b}$$

curvature tensors:

$$\begin{aligned} R_j{}^i{}_{mk} &= \frac{\partial F_j{}^i{}_k}{\partial x^m} - \frac{\partial F_j{}^i{}_k}{\partial y^r} N^r_m + F_j{}^r{}_k F_r{}^i{}_m - \frac{\partial F_j{}^i{}_m}{\partial x^k} + \frac{\partial F_j{}^i{}_m}{\partial y^r} N^r_k - F_j{}^r{}_m F_r{}^i{}_k + C_j{}^i{}_r R_k{}^r{}_m, \\ P_j{}^i{}_{mk} &= \frac{\partial C_j{}^i{}_m}{\partial x^k} - \frac{\partial C_j{}^i{}_m}{\partial y^r} N^r_k - C_r{}^i{}_m F_j{}^r{}_k + C_j{}^r{}_m F_r{}^i{}_k - C_j{}^i{}_r \frac{\partial N^r_k}{\partial y^m} - \frac{\partial F_j{}^i{}_k}{\partial y^m}, \\ S_j{}^i{}_{mk} &= \frac{\partial C_j{}^i{}_k}{\partial y^m} + C_r{}^i{}_m C_j{}^r{}_k - \frac{\partial C_j{}^i{}_m}{\partial y^k} - C_r{}^i{}_k C_j{}^r{}_m. \end{aligned} \tag{9}$$

The deflection tensors

$$D^i_k = F_0{}^i{}_k - N^i_k \quad \text{and} \quad C_0{}^i{}_k \tag{10}$$

are defined by the equations

$$d\varepsilon^a = -\omega^a_b \varepsilon^b + z^{-1}{}^a_i (\delta y^i + D^i_k \, dx^k + C_0{}^i{}_k \delta y_k).$$

§ 4. Finsler connections on line-element manifolds

The positive homogeneity of a Finsler connection is a very important property from the standpoint of the geometry of line-element manifolds [3]. In the following we shall characterize the positive homogeneous Finsler connections with help of the torsion and curvature tensors.

Definition 1. (MATSUMOTO) We say that the Finsler connection (Γ, N) is *strictly positively homogeneous* if the local parameters $F_i{}^j{}_k$, $C_i{}^j{}_k$, N_k^j of the connection (Γ, N) satisfy the following conditions in any coordinate system:

a) $$C_i{}^j{}_k(x, y), \quad F_i{}^j{}_k(x, y), \quad N_k^j(x, y)$$

are homogeneous functions with respect to the variables $y^1, \ldots, y^n$ of degree -1, 0, 1, respectively,

b) $$C_i{}^j{}_0 = 0.$$

Definition 2. (cf.[3]). We say that the Finsler connection (Γ, N) is of *Varga type* if it is strictly positive homogeneous and the local parameters $F_i{}^j{}_k$, $C_i{}^j{}_k$, N_k^j of the connection (Γ, N) satisfy the following condition in any coordinate system:

$$N_k^j = F_0{}^j{}_k.$$

Theorem 1. *Let be N a positively homogeneous non-lienar connection. Then the Finsler connection (Γ, N) is strictly positively homogeneous if and only if the following identities are fulfilled*

$$0 = C_j{}^i{}_0 = S_j{}^i{}_{0l} = P_j{}^i{}_{0l}.$$

Proof. Since $C_j{}^i{}_0 = 0$ we can write

$$S_j{}^i{}_{0k} = \frac{\partial C_j{}^i{}_k}{\partial y^m} y^m - \frac{\partial C_j{}^i{}_m}{\partial y^k} y^m = \frac{\partial C_j{}^i{}_k}{\partial y^m} y^m - \frac{\partial C_j{}^i{}_0}{\partial y^k} + C_j{}^i{}_k = \frac{\partial C_j{}^i{}_k}{\partial y^m} y^m + C_j{}^i{}_k.$$

Hence from the identity $S_j{}^i{}_{0k} = 0$ it follows the homogeneity of degree -1 of the functions $C_j{}^i{}_k$.

The curvature tensor $P_i{}^j{}_{kl}$ can be written in the form

$$P_i{}^j{}_{kl} = C_i{}^j{}_{k|l} + C_i{}^j{}_r \left(F_k{}^r{}_l - \frac{\partial N_l^r}{\partial y^k} \right) - \frac{\partial F_i{}^j{}_l}{\partial y^k},$$

where $C_i{}^j{}_{k|l}$ denotes the h-covariant derivative of the tensor $C_i{}^j{}_k$ *). It follows that

$$P_i{}^j{}_{0l} = C_i{}^j{}_{k|l} y^k + C_i{}^j{}_r \left(F_0{}^r{}_l - \frac{\partial N_l^r}{\partial y^k} y^k \right) + \frac{\partial F_i{}^j{}_l}{\partial y^m} y^m =$$

$$= C_i{}^j{}_{0|l} - C_i{}^j{}_k (y^k{}_{|l}) + C_i{}^j{}_r \left(F_0{}^r{}_l - \frac{\partial N_l^r}{\partial y^k} y^k \right) + \frac{\partial F_i{}^j{}_l}{\partial y^m} y^m.$$

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*) $T^i_{j|k} = \dfrac{\partial T^i_j}{\partial x^k} - \dfrac{\partial T^i_j}{\partial y^m} N_k^m + T_j^m F_m{}^i{}_k - T^i_m F_j{}^m{}_k$ for instance.

Since $C_i{}^j{}_0=0$, the functions N^i_j are homogeneous and $y^k{}_{|l}=F_0{}^k{}_l-N^k_l$, we get

$$P_i{}^j{}_{0l}=\frac{\partial F_i{}^j{}_l}{\partial y^m}\,y^m,$$

and thus from $P_i{}^j{}_{0l}=0$ it follows the homogeneity of degree 0 of the functions $F_i{}^j{}_l$.

The inverse part of the theorem is obvious.

Theorem 2. *The Finsler connection (Γ, N) is of Varga type if and only if the following identities are satisfied:*

$$0=D^i_k=C_j{}^i{}_0=S_j{}^i{}_{0l}=P_j{}^i{}_0=0.$$

Proof. The identity $S_j{}^i{}_{0l}=0$ is equivalent to the homogeneity of the functions $C_j{}^i{}_k$ as above.

Since $D^i_k=0$ we have

$$0=P_j{}^i{}_0=F_0{}^i{}_j-\frac{\partial N^i_j}{\partial y^k}\,y^k=N^i_j-\frac{\partial N^i_j}{\partial y^k}\,y^k,$$

that is the functions N^i_j are homogeneous of degree 1. As a consequence of $D^i_k=0$ we get the homogeneity of the functions $F_i{}^j{}_k$ too.

The inverse statement is obvious.

§ 5. On the equivalence problem

Let be given two Finsler connections by the connection parameters $\{N^i_k,\ F_j{}^i{}_k,\ C_j{}^i{}_k\}$ and $\{\tilde N^a_c,\ \tilde F_b{}^a{}_c,\ \tilde C_b{}^a{}_c\}$ in a coordinate neighborhood U in the manifold M with coordinates $x^1, \dots, x^n$. We say that these Finsler connections are locally equivalent if there exists an invertible point transformation $x^i=x^i(\bar x^1, \dots, \bar x^n)$ which carries the one connection into the other, i.e.

$$\tilde N^a_c=N^i_k\frac{\partial \bar x^a}{\partial x^i}\frac{\partial x^k}{\partial \bar x^c}+\frac{\partial \bar x^a}{\partial x^i}\frac{\partial^2 x^i}{\partial \bar x^c\,\partial \bar x^d}\,\bar y^d,$$

$$\tilde F_b{}^a{}_c=F_j{}^i{}_k\frac{\partial x^j}{\partial \bar x^b}\frac{\partial \bar x^a}{\partial x^i}\frac{\partial x^k}{\partial \bar x^c}+\frac{\partial^2 x^i}{\partial \bar x^b\,\partial \bar x^c}\frac{\partial \bar x^a}{\partial x^i},$$

$$\tilde C_b{}^a{}_c=C_j{}^i{}_k\frac{\partial x^j}{\partial \bar x^b}\frac{\partial \bar x^a}{\partial x^i}\frac{\partial x^k}{\partial \bar x^c},$$

where $x^1, \dots, x^n;\ y^1, \dots, y^n$ are the induced coordinates in TU, and $\bar x^a=\bar x^a(x^1, \dots, x^n)$ denotes the inverse point transformation of $x^i=x^i(\bar x^1, \dots, \bar x^n)$.

We can formulate the conditions of equivalence as follows.

Theorem 3. *The Finsler connections* $\{N^i_k, F_j{}^i{}_k, C_j{}^i{}_k\}$ *and* $\{\tilde{N}^a_c, \tilde{F}_b{}^a{}_c, \tilde{C}_b{}^a{}_c\}$ *given in a coordinate system* $x^1, \dots, x^n; y^1, \dots, y^n$ *are equivalent if and only if the following mixed system of differential equations is integrabile*

$$(11)\quad \text{(a)}\ \frac{\partial x^i}{\partial \bar{x}^a} = p^i_a,$$

$$\text{(b)}\ \frac{\partial y^i}{\partial \bar{x}^a} = p^i_c \tilde{N}^c_a - N^i_k p^k_a,$$

$$\text{(c)}\ \frac{\partial p^i_a}{\partial \bar{x}^b} = p^i_s \tilde{F}_a{}^s{}_b - F_k{}^i{}_l p^k_a p^l_b,$$

$$\text{(d)}\ \frac{\partial x^i}{\partial \bar{y}^a} = 0,$$

$$\text{(e)}\ \frac{\partial y^i}{\partial \bar{y}^a} = p^i_a,$$

$$\text{(f)}\ \frac{\partial p^i_a}{\partial \bar{y}^b} = 0,$$

$$(12)\qquad \tilde{C}_b{}^a{}_c p^i_a = \tilde{C}_k{}^i{}_l p^k_b p^l_c$$

for the unknown functions $x^i(\bar{x}^1, \dots, \bar{x}^n; \bar{y}^1, \dots, \bar{y}^n)$, $y^i(\bar{x}^1, \dots, \bar{x}^n; \bar{y}^1, \dots, \bar{y}^n)$, $y^i = y^i(\bar{x}^1, \dots, \bar{x}^n; \bar{y}^1, \dots, \bar{y}^n)$, $p^i_a = p^i_a(\bar{x}^1, \dots, \bar{x}^n; \bar{y}^1, \dots, \bar{y}^n)$, *where* $i, a = 1, \dots, n$, *and for the solution we have*

$$\det(p^i_a) \neq 0.$$

§ 6. The conditions of integrability

In the present section we shall determine the conditions of integrability of the mixed system (11), (12) ((11) $(a-f)$ are differential equiatons, (12) is a scalar relation) (cf. [1] pp. 14—18, [4] p. 73.).

It is easy to see that the commutation relations

$$\frac{\partial^2 x^i}{\partial \bar{x}^a \partial \bar{y}^b} = \frac{\partial^2 x^i}{\partial \bar{y}^b \partial \bar{x}^a}, \quad \frac{\partial^2 x^i}{\partial \bar{y}^a, \partial \bar{y}^b} = \frac{\partial^2 x^i}{\partial \bar{y}^b \partial \bar{y}^a}, \quad \frac{\partial^2 y^i}{\partial \bar{y}^a \partial \bar{y}^b} = \frac{\partial^2 y^i}{\partial \bar{y}^b \partial \bar{y}^a}$$

and $\dfrac{\partial^2 p^i_a}{\partial \bar{y}^c \partial \bar{y}^d} = \dfrac{\partial^2 p^i_a}{\partial \bar{y}^d \partial \bar{y}^c}$ are trivially satisfied.

Applying the equations (a) and (c) we can deduce from the commutation relation $\frac{\partial^2 x^i}{\partial \bar{x}^a \partial \bar{x}^b} = \frac{\partial^2 x^i}{\partial \bar{x}^b \partial \bar{x}^a}$ the condition

$$(13) \qquad p^i_c \tilde{T}_a{}^c{}_b = T_k{}^i{}_l p^k_a p^l_b.$$

From the relation $\frac{\partial^2 y^i}{\partial \bar{x}^a \partial \bar{y}^b} = \frac{\partial^2 y^i}{\partial \bar{y}^b \partial \bar{x}^a}$ we can be deduce

$$(14) \qquad p^i_c \tilde{P}_a{}^c{}_b = P_k{}^i{}_l p^k_a p^l_b$$

by using the equations (b), (c), (e), (f).

From the relation $\frac{\partial^2 p^i_a}{\partial \bar{x}^c \partial \bar{y}^b} = \frac{\partial^2 p^i_a}{\partial \bar{y}^b \partial \bar{x}^c}$ we derive the equation

$$(15) \qquad p^i_s \frac{\partial \tilde{F}_a{}^s{}_b}{\partial y^c} = \frac{\partial F_j{}^i{}_k}{\partial y^l} p^j_a p^k_b p^l_c$$

using (e) and (f).

From the commutation relation $\frac{\partial^2 y^i}{\partial \bar{x}^a \partial \bar{x}^b} = \frac{\partial^2 y^i}{\partial \bar{x}^b \partial \bar{x}^a}$ we get by the use of the equations (a), (b) and (c)

$$p^i_s \left(\tilde{F}_c{}^s{}_a \tilde{N}^c_b + \frac{\partial \tilde{N}^s_b}{\partial \bar{x}^a} - \tilde{F}_c{}^s{}_b \tilde{N}^c_a - \frac{\partial \tilde{N}^s_a}{\partial \bar{x}^b} \right) - P_r{}^i{}_t p^r_c p^t_a \tilde{N}^c_b + P_r{}^i{}_t p^r_c p^t_b \tilde{N}^c_a +$$
$$+ N^i_k (p^k_s \tilde{T}_a{}^s{}_b - T_l{}^k{}_m p^l_a p^m_b) = R_k{}^i{}_m p^m_a p^k_b.$$

Now we apply the relations (13) and (14), so we have

$$(16) \qquad p^i_c \tilde{R}_a{}^c{}_b = R_k{}^i{}_l p^k_a p^l_b.$$

At the end the commutation relation $\frac{\partial^2 p^i_a}{\partial \bar{x}^c \partial \bar{x}^b} = \frac{\partial^2 p^i_a}{\partial \bar{x}^b \partial \bar{x}^c}$ yields the equation

$$(17) \qquad p^i_s \tilde{T}_a{}^s{}_{bc} = T_k{}^i{}_{lm} p^k_a p^l_b p^m_c,$$

using (b), (c) and the relations (13), (15), where

$$T_k{}^i{}_{lm} \overset{\text{def}}{=} \frac{\partial F_k{}^i{}_l}{\partial x^m} - \frac{\partial F_k{}^i{}_l}{\partial y^r} N^r_m + F_r{}^i{}_m F_k{}^r{}_l - \frac{\partial F_k{}^i{}_m}{\partial x^l} + \frac{\partial F_k{}^i{}_m}{\partial y^r} N^r_l - F_r{}^i{}_l F_k{}^r{}_m$$

is the Varga's main curvature tensor (cf. [4], p. 13.).

We proved the following

Theorem 4. *A necessary and sufficient condition for the equivalence of two Finsler connections*

$$\{F_j{}^i{}_k, C_j{}^i{}_k, N^i_k\} \quad \textit{and} \quad \{\tilde{F}_b{}^a{}_c, \tilde{C}_b{}^a{}_c, \tilde{N}^a_c\} \quad \textit{is}$$

the existence of a whole number N such that the first N sets of equations in the sequence of sets of equations, which embody the laws of transformations (13)—(17) *of the tensors*

$$C_i{}^j{}_k,\quad T_i{}^j{}_k,\quad P_i{}^j{}_k,\quad \frac{\partial F_i{}^j{}_k}{\partial y^l},\quad R_i{}^j{}_k \quad \textit{and} \quad T_i{}^j{}_{kl} \quad \textit{and}$$

of the succesive h-covariant derivatives and ordinary derivatives by y^r, *shall be compatibile equations for the variables* x^i, y^i, p^i_j *as functions of the independent variables* $\bar{x}^a$, $\bar{y}^a$, *and all solutions of these equations shall satisfy the* $(N+1)$*st set of equations in the sequence.*

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On hereditarily finitely based varieties of semigroups

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Dedicated to Prof. H. Grell on his 70th birthday

In the investigation of the lattice of semigroup varieties one of the basic questions is the description of those varieties which have the property that all their subvarieties are finitely based. In particular, this means finding finite systems of identities which cannot be extended to infinite independent systems. P. PERKINS [4] has shown that commutativity has this propety and the author of the present paper has extended this result to a large class of permutative identities [5] (as L. SHEVRIN informed me recently the same generalization had been obtained by A. AĬZENŠTAT).*) However, E. LYAPIN has disappointed those who had hoped to go far by this way: in [3] he has shown that "most" balanced identities can be included in infinite independent systems. Here we continue the work in this direction: we are going to show that semigroup identities which define hereditarily finitely based varieties belong to a few exceptional types.

§ 1. The main results

Consider the free semigroup F and the free monoid F^0 over a countably infinite alphabet $X=\{x_i | i=1,2,\dots\}$. The elements of X will be occasionally denoted also by x, y, z, y_i, z_i. A *word* is an element of F^0, its identity element being the empty word $\emptyset$. If $u, v \in F^0$ and there exist two further words u', u'' such that $v=u'uu''$ then u is a *part* of v. If $u'=\emptyset$ or $u''=\emptyset$ then u is a *beginning part* or an *end part*, respectively.

Define the quasi-order $\lhd$ on F by

Definition 1. $u \lhd v$ $(u, v \in F)$ iff there exists an endomorphism $\varphi\colon F \to F$ such that $u\varphi$ is a part of v.

If we denote the length of a word w by $l(w)$ then $u \lhd v$ obviously implies $l(u) \leqq l(v)$. — Now we extend the relation $\lhd$ to subsets of F.

*) Remark at sheet-proof: As a matter of fact, this result has been obtained already by PUTCHA and YAQUB (*Semigroup Forum*, 3 (1971), 68—73) although they have not formulated it explicitly.

Definition 1′. $U \lhd V$ $(U, V \subseteqq F)$ iff $u \lhd v$ for at least one pair $(u, v) \in U \times V$. In this case we say that V depends on U; in the oppositive case it is *independent from U*.

Note that for subsets the relation $\lhd$ is not a quasi-order. — If $U = \{u\}$ we shall write also $u \lhd V$.

Definition 2. The set $V \subseteqq F$ is *dependent* (in itself) if $v \lhd V \setminus \{v\}$ for some $v \in V$, and *independent* in the opposite case.

Definition 3. A finite subset $U \subset F$ is an *essentially finite* set (EFS) if $U \lhd V$ holds for every infinite independent $V \subseteqq F$.

The significance of these notions for our purpose is established by the following propositions.

Proposition 1. *If the (finite or infinite) system of non-trivial semigroup identities*

$$(\tau) \qquad v_{2k-1} = v_{2k} \qquad (k = 1, 2, \ldots)$$

is such that $V = \{v_i | i = 1, 2, \ldots\}$ is an independent set of words then (τ) is an independent system of identities.

Proof. Denote by K_i the characteristic ideal of F generated by all v_j's, $j \neq i$, i.e. the ideal generated by $\{v_j \varphi | j \neq i, \varphi \in \mathrm{End}\,(F)\}$. It is easy to see that for every homomorphism $\chi\colon F \to F/K_i$ we have $v_j \chi = 0$ for $j \neq i$ so that all identities of (τ) but the one containing v_i hold in F/K_i. On the other hand, the independence of V means that $v_i \notin K_i$ so that the element $\overline{v_i}$ of F/K_i corresponding to v_i under the natural homomorphism is not 0. Thus, no identity in (τ) follows from the others.

Proposition 2. *Let (τ) be as in Proposition* 1 *but infinite and*

$$(\sigma) \qquad u_s = u_s' \qquad (s = 1, \ldots, m)$$

arbitrary. If $V = \{v_i | i = 1, 2, \ldots\}$ is independent from $U = \{u_i, u_i | 1 \leqq i \leqq m\}$ then $(\sigma) \cup (\tau)$ has no finite basis.

Indeed, as above, none of the identities in (τ) follows from the rest of $(\sigma) \cup (\tau)$. However, if a finite basis existed, it could be chosen as a subsystem of $(\sigma) \cup (\tau)$.

Corollary. *If $\mathfrak{S}$ is a hereditarily finitely based variety of semigroups and (σ) is a basis of $\mathfrak{S}$ then. U is an* EFS.

Indeed, in the opposite case an infinite independent system V would exist such that U non $\lhd V$ and the subvariety of $\mathfrak{S}$ defined by $(\sigma) \cup (\tau)$ would have no finite basis in virtue of Proposition 2.

By this Corollary, if we succeeded in determining all essentially finite subsets of F we could attain a considerable restriction of the scope of varieties which may be hereditarily finitely based. In this paper we determine all EFS's with 1 or 2 elements.

It holds obviously:

Proposition 3. *If $U \subseteqq U'$ and U is an* EFS *then so is U'.*

More generally,

Proposition 4. *If U is an* EFS *and U' is a finite set of words such that $U' \lhd \{u\}$ for every $u \in U$ then U' is an* EFS, *too.*

Indeed, every set depending on U depends on U'; so do all infinite independent sets.

In what follows we need some further definitions.

Definition 4. Two words u, v are *relatively prime* if no letter occurs in both of them.

Definition 5. u is a *closed part* of v if $v = u'uu''$ and u is relatively prime to u', u''. If $u \in X$ it is said to be a *closed letter.*

Definition 6. The decomposition $v = v_1 \dots v_k$ is *closed* if every pair v_i, v_j $(i \neq j)$ is relatively prime. (Notation: $v \doteq v_1 \dots v_k$.)

Definition 7. A *type* T is a subset of F consisting of all automorphic images of any of its own elements.

In other words T consists of all elements of F which differ from each other only by the notation of letters. If $u \in T$ we shall write also $T = T(u)$.

Definition 8. A word is *simple* if all its letters are closed (i.e. if they are all distinct). Denote the set of nonempty simple words by X^*.

Definition 9. Let T be a type. The word v is *T-simple* if $v \doteq \prod_{i=1}^{n} v_i$ where $v_i \in T \cup X^*$. Denote the set of all nonempty T-simple words by T^*.

In what follows, for $U \subseteq F$ we shall denote $U \cup \{\emptyset\}$ by U^0. Put furthermore $T_0 = \left\{v \mid v \doteq \prod_{i=1}^{k} v_i,\ v_i \in T\right\}$ and $T_1 = T^* \setminus T_0$.

Now we formulate the main results of our paper.

Theorem 1. *The one-element set $\{u\}$ is an* EFS *iff $u \in T(xyx) \cup X^*$.*

Theorem 2. *The two-element set U is an* EFS *iff its elements are of type T and T', respectively, where one of the following cases holds* (v, v', w, w' *always denote closed parts of the corresponding words, $w, w' \in X^{*0}$*):

(a)	$T = T(w)$,	T' *arbitrary*,	$w \neq \emptyset$;
(b)	$T = T(xyx)$,	T' *arbitrary*;	
(c)	$T = T(xwx)$,	$T' = T(x_1 y_1 x_1 w' x_2 y_2 x_2)$,	$w, w' \neq \emptyset$;
(d)	$T = T(xyzx)$,	$T' = T(vwv')$,	$w \neq \emptyset,\ v, v' \in T(x^2)^0$;
(d′)	$T = T(xyzx)$,	$T' = T(x^2)$;	
(e_1)	$T = T(xyzx)$,	$T' = T(x^2 wyzy)$;	
(e_2)	$T = T(xyzx)$,	$T' = T(yzywx^2)$;	
(f)	$T = T(wxyxw')$,	$T' = T(v)$,	$v \in T^*(x^2)$;

(g$_1$) $T=T(xyxw)$, $T'=T(xvyx)$, $v\in T^*(x^2)$;
(g$_2$) $T=T(wxyx)$, $T'=T(xyvx)$, $v\in T^*(x^2)$;
(h$_1$) $T=T(xyxz)$, $T'=T(xvx)$, $v\in T_1(x^2)$;
(h$_2$) $T=T(zxyx)$, $T'=T(xvx)$, $v\in T_1(x^2)$.

§ 2. Only if

In order to prove the "only if" parts we are going to list eight infinite independent word sets (Proposition 5); we shall see that all one- and two-element sets which the infinite subsets of the sets $V_1, \dots, V_6''$ depend on are those mentioned in the theorems. — For not to be obliged to prove the independence of each V_i separately we shall use the following

Lemma. *Let u, u' and w be words having the following properties:*

i) *their first letters coincide with the last ones;*
ii) $l(w)>1$;
iii) *if* $l(u)>1$ $(l(u')>1)$ *then* u non $\lhd$ w (u' non $\lhd$ w);
iv) *if* $l(u)=1$ $(l(u')=1)$ *then the letter* $u(u')$ *occurs in* $u'(u)$.

Then the set

$$V=\left\{v_n \middle| v_n = u\left(\prod_{i=1}^{n} w_i\right)u',\ w_i\in T(w),\ w_i \text{ closed}\right\}$$

is independent.

Sketch of the proof. Suppose $v_n\varphi$ is a part of v_m for $n<m$. First one shows that $u\varphi=u$ and $u'\varphi=u'$ (this follows from i, iii, iv and the fact that w_i is closed); hence $v_n\varphi=v_m$ and $l(w_k\varphi)>l(w)$ for some $k\leqq n$. However, this is impossible by i, ii and the same fact as before.

Now we obtain immediately

Proposition 5. *Put* $a_n=\prod_{i=1}^{n} x_{2i-1}x_{2i}x_{2i-1}$, $b_n=\prod_{i=1}^{n} x_i^2$. *Then the sets*

$V_1 = \{x_1 \dots x_{n-1}\, x_n x_{n-1} \dots x_1 x_n \mid n=3, 4, \dots\}$,

$V_2 = \{y_1 z_1 z_2 y_1 a_n y_2 z_3 z_4 y_2 \mid n=1, 2, \dots\}$,

$V_3 = \{y a_n y \mid n=1, 2, \dots\}$,

$V_4 = \{y_1^2 a_n y_2^2 \mid n=1, 2, \dots\}$,

$V_5 = \{y_1 z_1 y_1 b_n y_2 z_2 y_2 \mid n=1, 2, \dots\}$,

$V_6 = \{y b_n y \mid n=1, 2, \dots\}$,

$V_6' = \{yzyb_n z \mid n=1, 2, \dots\}$,

$V_6'' = \{z b_n yzy \mid n=1, 2, \dots\}$

are independent.

Proof. For V_1 this is known [1], for the rest it follows from the Lemma.

Before reverting to our Theorems, we state some simple propositions which we shall use later without refering to them.

Proposition 6. *If* $u \in X^*$ *then* $u \lhd v$ *iff* $l(u) \leqq l(v)$.

Proposition 7. xyx non $\lhd$ v *iff* $v \in T^*(x^2)$.

Proposition 8. $v \in T^*(xyx)$ *iff* x^2 non $\lhd$ v, $xyzx$ non $\lhd$ v.

Propisition 9. *If* $u\varphi = v_1 v_2 v_3$, v_2 *closed, then* $u = u_1 u_2 u_3$, u_2 *closed,* $u_i \varphi = v_i$ $(i=1, 2, 3)$.

Now we can prove the "only if" parts of both theorems.

Necessity (Theorem 1). If $\{u\}$ is essentially finite and u is not simple then $u \lhd V_1$ implies x^2 non $\lhd$ u. Hence, by $u \lhd V_6$, u must be of the form xwx, w simple. However $u \lhd V_4$ implies $l(w)=1$.

Necessity. (Theorem 2). Suppose $U=\{u_1, u_2\}$ is a minimal EFS (i.e. $u_1, u_2 \notin$ $\notin T(xyx) \cup X^*$). Put first $x^2 \lhd u_2$. Then $U \lhd V_1$ implies x^2 non $\lhd$ u_1. By the same reason, no letter can occur in u_1 more than twice. Moreover, $u_1 \lhd V_1$, $u_1 \lhd V_2$ imply that at most (and then, by minimality of U, exactly) one letter may occur twice. Indeed, if x_i and x_j both occur twice in u_1 and say, the first occurance of x_i precedes that of x_j then either this latter precedes the second occurance of x_i ($u_1 = \dots x_i \dots x_j \dots$ $\dots x_i \dots$) and u_1 non $\lhd$ V_2 or else (i.e. if $u_1 = \dots x_i \dots x_i \dots x_j \dots x_j \dots$) u_1 non $\lhd$ V_1. Thus, $u_1 = w x_i w' x_i w''$ (w, w', w'' simple and closed). Furthermore $u_1 \lhd V_2$ entails $l(w') \leqq 2$ and, since $u_1 \lhd V_3$, we have $w = w'' = \emptyset$ if $l(w') > 1$. Hence $u_1 = x_i x_j x_k x_i$ or $u_1 \doteq w x_i x_j x_i w''$.

In the first case u_1 non $\lhd$ V_4, u_1 non $\lhd$ V_5 and so $u_2 \lhd V_4$ which implies $u_2 \in$ $\in T(x^2)$ — case (d') — or $u_2 \doteq v_1 q v_2$ with $v_1, v_2 \in T(x^2) \cup T(xyx) \cup X$, x^2 non $\lhd$ q, $q \neq \emptyset$ if $v_1, v_2 \in T(x^2)$, and $u_2 \lhd V_5$ which implies xyx non $\lhd$ q, $q \neq \emptyset$ if $v_1, v_2 \in T(xyx)$. Now x^2 non $\lhd$ q, xyx non $\lhd$ q give $q \in X^{*0}$ and we get either one of the cases (d), (d'), (e_1), (e_2) or a subcase of (g_1) or (g_2).

If $u_1 \doteq w x_i x_j x_i w'$ then $w = w' = \emptyset$ is impossible in virtue of the minimality of U. Suppose $w \neq \emptyset$. Then u_1 non $\lhd$ V_6, $u_2 \lhd V_6$ and either $u_2 \in T^*(x^2)$ and we obtain case (f) or $u_2 = x_{m+1} v x_{m+1}$, $v \in T^*(x^2)$, v closed. In the latter case u_2 non $\lhd$ V_5, $u_1 \lhd V_5$; hence $w' = \emptyset$. Furthermore, if $v \in T_0(x^2)$, $l(v) = 2m$ then u_2 non $\lhd$ $V_5^{(m)} =$ $= \{y b_n y \mid n > m\}$. Thus, $v \in T_1(x^2)$. Now if $l(w) = 1$ we have case (h_2); if $l(w) > 1$ then u_1 non $\lhd$ V_6, $u_2 \lhd V_6$ which implies $v \doteq y v'$ and we obtain (g_2). The case where $w' \neq \emptyset$ can be settled analogously.

Now let x^2 non $\lhd u_1$, x^2 non $\lhd u_2$. Put $u_1 \lhd V_6$; then $u_1 = x_{m+1} w x_{m+1}$, $w \in X^*$, closed, $l(w) > 1$ by the minimality of U. Now u_1 non $\lhd V_5$, and thus $u_2 \doteq wu'w'$ where $w, w' \in T(xyx)^0$, $u' \in X^*$. If $w, w' \neq \emptyset$ we have case (c), if $w = \emptyset$ or $w' = \emptyset$ we get a subcase of (g_1) or (g_2), respectively. This completes the proof of the necessity.

§ 3. If

In proving that the sets given in Theorems 1 and 2 are essentially finite we have to show that infinite sets depending on them are dependent in themselves. For this, we need some theorems which assure the dependence of certain types of infinite sets, and these theorems in their turn are based on some results in the theory of q.o. sets. We are going now to quote these latter ones.

Definition 10. The quasi-ordered set P is a *well quasiordered* set (WQOS) if it satisfies the descending chain condition and does not contain infinite independent subsets (i.e. infinite sets of pairwise incomparable elements).

Next we give some plain facts.

(I) *Let P be a quasi-ordered set. If there exists a mapping γ of P in a* WQOS *R such that $p\gamma \leqq p'\gamma$ implies $p \leqq p'$ then P is a* WQOS *itself.*

Let us mention two important particular cases:

(I_1) *A subset of a* WQOS *is a* WQOS.

(I_2) *If $\prec$ is a refinement of the q.o. $<$ on P and P is a* WQOS *under $<$ then so it is under $\prec$.*

(II) *The union of a finite number of* WQOS's *is a* WQOS.

Now let P be a q.o. set. Define a q.o. on the set $\bar{P}$ of all finite sequences of elements of P by

$$\pi = (p_1, \dots, p_n) \leqq (p'_1, \dots, p'_m) = \pi'$$

iff there exists a subsequence $(p'_{i_1}, \dots, p'_{i_n})$, $1 \leqq i_1 < \dots < i_n \leqq m$, of π' such that $p_j \leqq p'_{i_j}$. The following proposition is a consequence of a theorem of G. Higman [2].

(III) *If P is a* WQOS *then so is $\bar{P}$.*

We prefer to give here a self-contained proof. First of all, the classical theorem of Ramsey implies:

(IV) *The direct product of a finite number of* WQOS's *is a* WQOS.

It is routine to check the validity of the descending chain condition in $\bar{P}$. Now suppose P_1 is an infinite independent subset of $\bar{P}$ and let $\pi = (p_1, \dots, p_n)$ be an element of minimal length in P_1. Suppose $\pi' = (p_1, \dots, p_{k-1})$ is the maximal segment of π such that the subset $R = \{\varrho \in P_1 \mid \pi' < \varrho\}$ of P_1 is infinite; obviously, $0 < k \leqq n$. Choose a subsequence $\varrho'=(r_{\varrho(1)}, \ldots, r_{\varrho(k-1)})$ in each $\varrho\in R$ such that $p_i \leqq r_{\varrho(i)}$, each $\varrho(i)$ as small as possible. By (IV) and the theorem of RAMSEY there exists an infinite subset R' of R such that the set $R''=\{\varrho' \mid \varrho\in R'\}$ is totally quasi ordered (i.e. any two elements of R'' are comparable).

Now we "break up" the elements of R': for each $\varrho\in R'$ put $\varrho^{(i)}=\{r_{\varrho(i-1)+1}, \ldots, r_{\varrho(i)-1}\}$ where $\varrho(0)=0$, $\varrho(k)=l(\varrho)+1$ and $\varrho(i)$ is defined as above if $1\leqq i\leqq k-1$. It can be seen easily that every component of $\varrho^{(i)}$ $(i=1, \ldots, k)$ is either strictly less then p_i or incomparable with it. At least one of the sets $R^{(i)}=\{\varrho^{(i)} \mid \varrho\in R'\}$ must contain an infinite independent subset; indeed, in the opposite case all they are WQOS's and, by (IV), so is the direct product $Q=R''\oplus R^{(1)}\oplus\ldots\oplus R^{(k)}$. However then the mapping $\gamma: R'\to Q$ $(\varrho\gamma=(\varrho', \varrho^{(1)}, \ldots, \varrho^{(k)}))$ satisfies the conditions in (I) and therefore R' is a WQOS which is a contradiction.

Suppose $R^{(i_1)}$ contains an infinite independent subset P_2. Repeating the above construction, we find a component $p_{i_1 i_2}$ of a vector of minimal length in P_2 such that there exists an infinite independent set P_3 consisting of vectors with components stricly less than or incomparable with $p_{i_1 i_2}$. Thus we obtain an infinite series p_{i_1}, $p_{i_1 i_2}, \ldots$ of elements of P having the property that every member of it is either strictly less than or incomparable with each of the preceding ones which is impossible.

In applying these facts to word sets we shall use besides $\lhd$ three further relations:

Definition 1_l. $u\lhd_l v$ iff there exists an endomorphism $\varphi: F\to F$ such that $u\varphi$ is an end part of v $(v=v'\cdot u\varphi)$.

Definition 1_r. $u\lhd_r v$ iff there exists an endomorphism $\varphi: F\to F$ such that $u\varphi$ is a beginning part of v $(v=u\varphi\cdot v')$.

Definition 1_q. $u\lhd_q v$ iff there exists an endomorphism $\varphi: F\to F$ such that $v=u\varphi$.

For a set of words V the q.o. set $\{V, \lhd\}$ will be simply denoted by V; furthermore, we shall write
$$V_l=\{V, \lhd_l\},\quad V_r=\{V, \lhd_r\},\quad V_q=\{V, \lhd_q\}.$$
Obviously, all these sets satisfy d.c.c. By (I_2), if V_q is a WQOS then so is V_l and V_r and if either of these latter ones is a WQOS then so is V.

We introduce an operation $\circ$ on sets of words as follows:
$$U\circ V=\{w \mid w\doteq uv,\ u\in U,\ v\in V\}.$$
It holds

Theorem 3. *Let* $U, U', V_r, V_l', W_q, W_q'$ *be* WQOS's. *Then* $V'\circ V$, $(V'\circ W)_l$, $(W'\circ V)_r$ *and* $(W'\circ W)_q$ *are* WQOS's. *Moreover, if either the last letter of every element of* U *and* V *or the first letter of every element of* U' *and* V' *is closed then* $U\circ U'$, $(U\circ W')_l$. $(W\circ U')$, *and* $(V\circ V')_q$ *are* WQOS's, *too.*

Proof. Let us prove the assertion concerning $U \circ U'$; in the other cases the proof runs analogously. Suppose for example that $U = \{u_i | u_i \doteq u_i^* y_i,\ i = 1, 2, \ldots\}$, $U' = \{u'_j | j = 1, 2, \ldots\}$. Then $U \circ U' = \{w_{ij} | w_{ij} \doteq u_i u'_j\}$. Now the direct product $U \times U'$ is a WQOS by (IV). Consider the mapping $\gamma: U \circ U' \to U \times U'$ defined by $w_{ij}\ \gamma = (u_i, u'_j)$. Now let $(u_i, u'_j) \leqq (u_k, u'_p)$ i.e. $u_i \lhd u_k$, $u'_j \lhd u'_l$ and, say, $u_k = s(u_i \varphi)t = s(u_i^* \varphi)(y_i \varphi)t$, $u'_l = s'(u'_j \psi)t'$. As u_i^* and u'_j are relatively prime and none of them contains y_i, there exists an endomorphism $\chi: F \to F$ such that $u_i^* \chi = u_i^* \varphi$, $u'_j \chi = u'_j \psi$ and $y_i \chi = y_i \varphi \cdot ts'$. Thus $w_{kl} = u_k u'_l = s w_{ij} \chi t'$, i.e. $w_{ij} \lhd w_{kl}$. Hence the assertion follows by (I).

Theorem 4. *Let T be a type. Then T^*, T_l^*, T_r^*, T_{1q} are* WQOS's.

Proof. First remark that X_q^* and T_{0r} are WQOS's. Hence, by Theorem 3 (case $(V \circ V')_q$) and (II), the posets $T'_q = (T_0 \circ X^*)_q$ and $T''_q = (T' \cup X^*)_q$ are WQOS's. Using the notation $T'^* = \left\{w | w = \prod_{i=1}^{n} t_i,\ t_i \in T'\right\}$, we have $T_1 = X^* \cup T'^* \cup (X^* \circ T'^*) \cup \cup (T'^* \cup (X^* \circ T'^*)) \circ T_0$. Thus, it suffices to show that $T_l'^*$ is a WQOS since then by Theorem 3 $(T' \circ T'^*)_q$ and hence $(T' \cup (T' \circ T'^*))_q = T_q'^*$ are also WQOS's. Using again Theorem 3, (I_2) and (II) we conclude, furthermore, that T_{1q} and $T_r^* = (T_0 \cup T_1)_r$ are WQOS's, too. The rest follows by duality and by (I_2), respectively.

Since T'_q is a WQOS, the same follows from (III) for the set of finite sequences $\overline{T'_q}$. Consider the mapping $\gamma: T_l'^* \to \overline{T'_q}$ defined by $w\gamma \doteq \left(\prod_{i=1}^{n} t_i\right)\gamma = (t_1, \ldots, t_n)$ $(t_i \in T')$. We have to show that γ satisfies the condition of (I). Put $w' = \prod_{j=1}^{m} t'_j \in T'^*$ and suppose $(t_1, \ldots, t_n) \leqq (t'_1, \ldots, t'_m)$ in $\overline{T'_q}$, i.e. $t_i \lhd_q t'_{j_i}$ for some $1 \leqq j_1 < \ldots < j_n \leqq m$. In other words, there exist endomorphisms $\varphi_i: F \to F$ $(i = 1, \ldots, n)$ such that $t_i \varphi_i = t'_{j_i}$. Denote the last letter of t_i by y_i; then $t_i \doteq t_i^* y_i$. Since $t_1^*, \ldots, t_n^*$ are pairwise relatively prime and they do not contain $y_1, \ldots, y_n$, there exists an endomorphism $\varphi: F \to F$ such that $t_i^* \varphi = t_i^* \varphi_i$ for $i = 1, \ldots, n$; $y_i \varphi = y_i \varphi_i \cdot \prod_{k=j_i+1}^{j_{i+1}-1} t'_k$ for $i = 1, \ldots, n-1$ and $y_n \varphi = y_n \varphi_n \prod_{k=j_n+1}^{m} t'_k$. Hence $t_i \varphi = \prod_{k=j_i}^{j_{i+1}-1} t'_k$ for $i < n$, $t_n \varphi = \prod_{k=j_n}^{m} t'_k$ and $w' = \left(\prod_{k=1}^{j_1-1} t'_k\right) \times \times (w\varphi)$. The theorem is proved.

Theorem 5. *Let $V = \{v_i \mid i = 1, 2, \ldots\}$ be a set of words. Suppose there exist natural numbers k, l, n and n types $T^{(1)}, \ldots, T^{(n)}$ such that every v_i has a decomposition $v_i = v_{i0} \cdot \prod_{j=1}^{n} u_{ij} v_{ij}$ where*

a) $l(u_{ij}) \leqq l$,

b) $v_{ij} \in T^{(j)*0}$,

c) v_{ij} *is closed if non-empty,*

d) *if* $l(v_{ij}) > k$ *for some* $j \neq 0, n$ *then* $v_{ij} \in T_1^{(j)}$.

Then V is dependent. Moreover, if $v_{i0}\neq\emptyset$ ($v_{in}\neq\emptyset$) for every i then V_r (V_l) is dependent as well.

Proof. We shall prove that V contains an infinite WQOS. Indeed, there is only a finite number of sequences $u_1, \ldots, u_n$ of words of bounded length. Consequently, V has an infinite subset V' such that to every $v_i\in V'$ the same sequence $u_1, \ldots, u_n$ corresponds. Now $V'=V_0^*\cup V_0^{**}$ where $V_0^*=\{v_i \mid v_i\in V',\ v_{i0}\neq\emptyset\}$, $V_0^{**}=V'\setminus V_0^*$. Put $V_0'=V_0^*$ if V_0^* is infinite and $V_0'=V_0^{**}$ in the opposite case. Construct in a similar way consecutively $V_0'\supseteqq V_1'\supseteqq\ldots\supseteqq V_n'$ with either $V_r'=V_r^*=\{v_i \mid v_i\in V_{r-1}',\ v_{ir}\neq\emptyset\}$ or $V_r'=V_{r-1}'\setminus V_r^*$. For sake of simplicity suppose $u_j\neq\emptyset$ and, for $j\neq 0$, $j\neq n$, $V_j'=V_j^*$ (in the opposite case we possibly had to change the parameters k, l, n and the decomposition of v_i). Put $T_0^{(j)}(k)=\{w \mid w\in T_0^{(j)},\ l(w)\leqq k\}$, $U_j=T_1^{(j)}\cup T_0^{(j)}(k)$ for $1\leqq j\leqq n-1$, $U_0=T^{(0)*}$ or $\{\emptyset\}$ according to $V_0'=V_0^*$ or V_0^{**} and $U_n=T^{(n)*}$ or $\{\emptyset\}$ according to $V_n'=V_n^*$ or V_n^{**}. Then $v_{ij}\in U_j$ for $v_i\in V_n'$, and U_{jq} ($1\leqq j\leqq n-1$) as well as U_{0l}, U_{nr} (if different from $\{\emptyset\}$) are WQOS's. This implies that $A=U_{0l}\times\times\left(\prod_{j=1}^{n-1} U_{jq}\right)\times U_{nr}$ is also a WQOS. Define $\gamma\colon V_n'\to A$ by $v_i\gamma=\left(v_{i0}\cdot\prod_{j=1}^{n} u_j v_{ij}\right)\gamma==(v_{i0}, \ldots, v_{in})$. Suppose $v_h\in V_n'$ and $(v_{i0}, \ldots, v_{in})\leqq(v_{h0}, \ldots, v_{hn})$ in A, i.e. $v_{h0}==w\cdot(v_{i0}\varphi_0)$, $v_{hn}=(v_{in}\varphi_n)\cdot w'$ (if non-empty) and $v_{hj}=v_{ij}\varphi_j$ ($j=1, \ldots, n-1$) with some suitable endomorphisms $\varphi_0, \ldots, \varphi_n$ of F. In consequence of c), there exists $\varphi\colon F\to F$ such that $u_j\varphi=u_j$, $v_{ij}\varphi=v_{ij}\varphi_j$ so that $v_h=w(v_i\varphi)w'$. Consequently, V_n is an infinite WQOS and therefore contains comparable elements which completes the proof.

Now we are in position to prove the second parts of Theorems 1 and 2.

Sufficiency (Theorem 1). In consequence of Proposition 6, every infinite set depends on $u\in X^*$. If $u\in T(xyx)$ and u non $\lhd V$ then, by Proposition 7, $V\subseteqq T^*(x^2)$ and it is dependent in itself by Theorem 4. This completes the proof of Theorem 1.

Sufficiency (Theorem 2). We shall show that U is an EFS in cases (c), (e_1), (f), (g_1), (h_1) and in the subcase of (d) where $v, v'\in T(x^2)$. Cases (e_2), (g_2), (h_2) follow then by duality, (d') and the rest of (d) from Proposition 4, and (a), (b) from Proposition 3. In all cases the proof consists in finding the general form of words which are independent from U and in a subsequent application of Theorem 5 to infinite sets consisting of such words. Thus, put U non $\lhd q$. If $q=q_1xq_2$ we shall say that x occurs in q later (earlier) again if $q_2=q'xq''$, $q'\neq\emptyset$ ($q_1=q'xq''$, $q''\neq\emptyset$).

Case (c). Let $q\doteq q'tq''$ where $q', q''\in T^*(x^2)^0$ and either $t=\emptyset$ or the first letter of t occurs later again, the last one earlier again in q. We have $l(t)<2l(w)+l(w')+2$. Indeed, the number of letters between the first and last occurrance of the extreme letters of t cannot exceed $l(w)-1$, and the number of those between the last occurrance of the first letter of t and the first occurance of the last one must be less than $l(w')$. Put k arbitrary, $l=2l(w)+l(w')+1$, $n=1$, $T^{(0)}=T^{(1)}=T(x^2)$ and apply Theorem 5.

Case (d), $v, v' \in T(x^2)$. Let $q \doteq q'tq''$ where $q'q'' \in T^*(xyx)^0$, $t = x_i^2 t' x_j^2$ or $t \in T(x^2)^0$ (as $xyzx$ non $\lhd$ q, such a decomposition exists by Proposition 8). Then $l(t') < l(w)$ and we can apply Theorem 5 with k arbitrary, $l = l(w)+3$, $n=1$, $T^{(0)} = T^{(1)} = T(xyx)$.

Case (e$_1$). Since $xyzx$ non $\lhd$ q, there exists a decomposition $q \doteq q'tq''$ where $q' \in T^*(xyx)^0$, $q'' \in T^*(x^2)^0$ and $t = x_i^2 t' x x_j x_k x_j$ (or $t = \emptyset$). Then $l(t') < l(w)$. Put k arbitrary, $l = l(w)+4$, $n=1$, $T^{(0)} = T(xyx)$, $T^{(1)} = T(x^2)$ and Theorem 5 yields the dependence.

Case (f). If $q = q_1 x_j q_2 x_j q_3$ with $q_2 \neq \emptyset$ then either $l(q_1) < l(w)$ or $l(q_3) < l(w')$. Hence $q = tq't'$ where $l(t) \leqq l(w)$, $l(t') \leqq l(w')$ and $q' \in T^*(x^2)^0$; moreover, if $x_{i_1}, \ldots, x_{i_s}$ are the letters which occur in q' and in either t or t', too, then $q = tv_0 \left(\prod_{j=1}^{s} x_{i_j}^{\varepsilon_j} v_j \right) t'$ where $\varepsilon_j = 1$ or 2, $v_j \in T^*(x^2)^0$, v_j closed if non-empty and if $l(v_j) \geqq 2m(v)$ $\big(m(v)$ denotes the number of different letters in $v\big)$ then $v_j \in T_1(x^2)$ (in the opposite case we had $v \lhd q$). Thus, we can apply Theorem 5 putting $k = 2m(v)$, $l = 2\big(l(w)+l(w')\big)$, some $n \leqq l(w)+l(w')+2$ and $T^{(0)} = \ldots = T^{(n)} = T(x^2)$.

Case (g$_1$). Here $q = v_0 \left(\prod_{j=1}^{s} x_{i_j}^{\varepsilon_j} v_j \right) t$ where $l(t) = l(w)$, $x_{i_1}, \ldots, x_{i_s}$ are different letters occurring also in t, $\varepsilon_j = 1$ or 2, $v_j \in T^*(x^2)^0$, v_j closed if non-empty and $v_j \in T_1(x^2)$ if $l(v_j) > 2m(v)$, $j \neq 0$ (in the opposite case $xvyx \lhd q$). One can apply Theorem 5 with $k = 2m(v)+1$, $l = 2l(w)$, some $n \leqq l(w)+1$ and $T^{(0)} = \ldots = T^{(n)} = T(x^2)$.

Case (h$_1$). Now either $q \in T^*(x^2)$ or $q = v_0 x_i v_1 x_i$ where $\varepsilon = 1$ or 2, $v_0, v_1 \in T^*(x^2)^0$, closed, and $v_1 \in T_1(x^2)$ if $l(v_1) > 2m(v)$. Again, put $k = 2m(v)$, $l = 2$, $n = 2$, $T^{(0)} = T^{(1)} = T^{(2)} = T(x^2)$ (as a matter of fact, $T^{(2)}$ is unrelevant since $U_2 = \emptyset$).

Theorem 2 is proved.

Corollary. *If the identity $u = u'$ defines a hereditarily finitely based variety then the pair of types $T(u)$, $T(u')$ is one of the pairs (a)—(h$_2$).*

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Bibliographie

Iain T. Adamson, Elementary Rings and Modules, (University Mathematical Texts) 136 pages, Edinburgh, Oliver and Boyd, 1972.

This book is an elementary introduction to some basic notions and ideas of commutative algebra. It is well-written and self-contained in the sense that it contains an explanation of the steadily used general algebraic concepts. Thus it is very suitable for undergraduate students as a textbook. Examples and exercises, a reading list and an index complete this small but useful book.

Béla Csákány (Szeged)

F. F. Bonsall and J. Duncan, Complete Normed Algebras (Ergebnisse der Mathematik und ihrer Grenzgebiete, 80), X+301 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1973.

The theory of Banach algebras has wide-ranging application in harmonic analysis, operator theory and function algebras. Moreover it disposes of a rich collection of general results some of which deserve to be known by every mathematician.

The present book is an excellent account of the principal methods and results in the theory of Banach algebras, both commutative and noncommutative. It is a new and indispensable source for anyone, student of researcher, working in this area. The highly developed theory of C*-algebras, function algebras and group algebras is outside the scope of this monograph as well as the multipliers, the extensions and other generalizations of Banach algebras.

Chapter 1: Concepts and Elementary Results. It deals with elementary facts on normed algebras, inverses, equivalent norms, the spectrum and contour integrals. An elegant exposition is given for the functional calculus, the elementary functions, the numerical range and the approximate identities. Normed division algebras conclude the chapter.

Chapter 2: Commutativity. It begins with the characterization of multiplicative linear functionals and the Gelfand representation theory of commutative Banach algebras. A concise and elegant account of derivations, joint spectra and the functional calculus for several elements succeeds. The final topic of this chapter is the theory of functions analytic on a neighbourhood of the carrier space, the Shilov boundary and the hull-kernel topology of that space.

Chapter 3: Representation Theory. After some algebraic preliminaries the continuity of the irreducible representations of a Banach algebra on normed spaces is discussed. Also, the structure space of a noncommutative algebra, the A-Module pairings, and the dual modules are treated here. The theory of representations of linear functionals concludes the chapter.

Chapter 4: Minimal ideals. The necessary algebraic arrangement is followed by the theory of annihilator algebras. Recent results on compact action on Banach algebras and the basic facts on H*-algebras are given.

Chapter 5: Star Algebras. The theory iniciated by Gelfand and Naimark of representation of star algebras by positive linear functionals is treated. Recent results on charachterizations of C*-algebras and B*-semi-norms and the new, nice theory of hermitian algebras conclude the chapter.

Chapter 6: Cohomology. Tensor products, amenable Banach algebras and the recent theory of cohomology of Banach algebras are treated.

Chapter 7: Miscellany. It begins with capacity and positiveness of the spectrum in Banach algebras. The theory of type O and locally compact semi-agebras is discussed. Recent results on characterization of Q-algebras conclude the book.

The presentation of the book attains an optimum in consiseness and readability. Summarizing, the autors did an important and beautiful job in writing this monograph.

Zoltán Sebestyén (Budapest)

André Delachet, La géométrie élémentaire, Le calcul vectoriel, Le calcul tensoriel, 128, 128, 128 pages, Presses Universitaires de France, Paris, 1966, 1967, 1969.

These little books were published in the series "Que sais-je?". No mathematical knowledge beyond the secondary school level is necessary to read them. They were written mostly for beginners of higher training, but they are also useful and well constructed introductions to these areas of mathematics for everybody interested in mathematics.

"La géometrie élémentaire" treats the axiomatic foundations of the geometry of the plane using the concept of set and real number. It follows the method of M. Choquet and gives a good insight into the axiomatic method.

"Le calcul vectoriel" deals with vectors in 3 dimensional physical space. After the definitions of the concept of vectors and the fundamental operations with them it discusses many physical applications: statics of the rigid solid, speed of the moving point in different systems of co-ordinates, the moving of a rigid solid. It also touches upon the bases of differential geometry.

"Le calcul tensoriel" supposes some knowledge in Linear Algebra. In the first part the tensor algebra (concept of tensor product, affine tensors, exterior algebra, Euclidian tensors) are treated. The second part deals with tensor fields defined in Euclidian and Riemannian spaces and their covariant derivatives.

L. Kérchy (Szeged)

W. W. Comfort, S. Negrepontis, Continuous Pseudometrics, 126 pages, New York, M. Dekker, Inc., 1975.

The book contains a detailed exposition of an interesting part of general topology. The topics concerned belong for a large part to what may be called, in a broad sense, the descriptive theory of sets in topological spaces. The title does not perhaps express the content quite fully, though pseudometrics are used quite often indeed, both explicitly and implicitly.

As the authors point out, the book is based on classroom notes elaborated during the academic year 1967—68. The exposition is clear and presupposes almost no knowledge of general topology.

In Section 1 the general concepts of P-embedding and P-fication are introduced for an arbitrary class P of completely regular spaces (if, e.g., P is the class of compact spaces, then Y is the Stone-Čech compactification of X if and only if Y is a P-fication of X and X is P-embedded in Y). Using these notions, the basic facts concerning paracompact, realcompact, topologically complete, etc., spaces are presented in a systematic and lucid way in Sections 2 through 7. A number of deeper results, e.g. Glicksberg' s theorem on the Stone-Čech compactification $\beta(X\times Y)$ are included.

In Section 8 Borel metrisable separable spaces are considered. Sections 9—11 contain some topics that are not quite current and seem to appear in a book for the first time. In Section 10, the local connectedness of βX is examined. In Section 9, the authors consider Baire sets and Baire spaces, defined as follows: X is Baire in $Y \supset X$ if X can be obtained from zero-sets (i.e. sets $f^{-1}0$ where $f: Y \to R$ is continuous) in countably many steps by taking countable unions and intersections. X is a Baire space if it is Baire in βX. Section 11 contains counterexamples.

The book is a useful exposition of important and interesting topics including also some less known results. It may well serve as a basis of a special course in general topology, or of a part of a more extensive introductory course.

M. Katětov (Prague)

Steven A. Gaal, Linear Analysis and Representation Theory (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 198), IX+688 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1973.

The book is an introduction to a number of topics in functional analysis, harmonic analysis and representation theory in Hilbert space. In the Preface the author says: "... I tried not to be encyclopedic but rather select only those parts of each chosen topic which I could present clearly and accurately in a formulation which is likely to last. The material I chose is all mathematics which is interesting and important both for the mathematician and to a large extent also for the mathematical physicist". The book is intended for "... frequent browsing, consultation and other occasional use". The exposition is clear and concise and proofs are as simple in concept as possible, so Gaal's work can serve as a textbook for students and also as a reference book. The chapters are made "as independently readable as possible under the given conditions". The chapter headings will give the reader of this review some idea of the book's content: Algebras and Banach Algebras; Operators and Operator Algebras; The Spectral Theorem, Stable Subspaces and v. Neumann Algebras; Elementary Representation Theory in Hibert Space; Topological Groups, Invariant Measures, Convolutions and Representations; Induced Representations; Square Integrable Representations, Spherical Functions and Trace Formulas; Lie Algebras, Manifolds and Lie Groups. A bibliography, subject index, and index of notations and special symbols are provided

J. Szűcs (Szeged)

I. C. Gohberg—M. G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators (Translations of Mathematical Monographs, Vol. 18), XV+378 pages, American Mathematical Society, Providence, R. I., 1969.

During the last 20—25 years the theory of non-selfadjoint operators in Hilbert space grew to an important branch of functional analysis, which has now its own methods and typical results. In the Soviet Union research in this direction was started by M. V. Keldyš, M. S. Livšic and L. A. Sahnovi , and intensively and on a very wide scale continued by M. G. Kreĭn, M. S. Brodskiĭ, I. C. Gohberg, V. I. Macajev and others. Their investigations can be characterized by intense use of results of complex function theory in connection with estimates of the resolvent of a given operator, with (infinite) perturbation determinants and characteristic functions. An essential role in these investigations is played by ideals of compact operators, typical results are e.g. statemants about the completeness of the system of root vectors of an operator or an operator pencil, canonical representations of the operator (wich can be considered as generalizations of the triangular representation of a quadratic matrix) and abstract factorizations.

This monograph (the original edition in Russian appeared in 1965) contains the foundations of this theory and its results about the mentioned completeness problems. After a first introductory chapter (here one finds e.g. Gohberg's theorem on holomorphic operator functions) the second chapter contains a systematic exposition of the theory of s-numbers (absolute eigenvalues) of a compact operator (e.g. the inequalities of H. Weyl, Ky Fan, A. Horn). They form a basis for a systematic study of the symmetrically-normed ideals of the ring of bounded linear operators in Hilbert space in the third chapter, where also — compared with the classical exposition by R. Schatten — many new results can be found. In the fourth chapter the theory of perturbation determinants and some of its applications are given. Especially important are here the connections of the hermitean components of a Volterra operator. These and other methods are used in the fifth chapter in order to prove deep results on the completeness of root vectors of certain classes of operators. Finally, in the last chapter some notions of a basis in a Hilbert space are discussed and bases of root vectors of certain dissipative operators are considered. Most of the material of the book is here for the first time exposed in a monograph, many of the proofs had not at all been published before. So the appearence of this book was one of the highlights in the theory of non-selfadjoint operators. During the past ten years it became a frequently quoted standard reference with great influence on the development of operator theory. In a certain sense it has a continuation in the book *Theory of Volterra operators in Hilbert space and its applications* by the same authors, where e.g. questions of canonical representations and factorizations are treated. These two books and the monograph by Sz.-Nagy and C. Foiaş: *Harmonic analysis of operators on Hilbert space* cover a great and essential part of what is nowadays known about non-selfadjoint operators on Hilbert space.

H. Langer (Dresden)

Nathan Jacobson, Basic Algebra. I, XVI+472 pages, San Francisco, W. H. Freeman and Co., 1974.

This book is a very attractive introduction to the abstract algebra for undergraduate students; it is also an excellent tool to refresh and supplement the knowledge of their reachers. The author of *Lectures in Abstract Algebra* hardly needs any praise for conciseness and clarity, but it is impossible to leave unmentioned his excellent ability, amply proved again by this book, to present algebra in a modern and gently simple style. This does not mean that there are merely easy facts dealt with here; e.g. the transcendence of π found room in the book as well as the description of several sequences of finite simple groups.

The volume consists of eight chapters (preceded by an introduction where some basic set-theoretic and arithmetic facts are summarized). To make feel its flavour, let us list them here: 1. Monoids and groups. 2. Rings. 3. Modules over a principal ideal domain. 4. Galois theory of equations. 5. Real plynomial equations and inequalities. 6. Metric vector spaces and the classical groups. 7. Algebras over a field. 8. Lattices and Boolean algebras.

Béla Csákány (Szeged)

N. S. Landkof, Foundations of modern potential theory (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 180), X+424 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1972.

This monograph on potential theory is a translation, by A. P. Doohovskoy, of the original in Russian: *Osnovy sovremennoĭ teorii potenciala*, Nauka (Moscow, 1966). At that time there was no monograph on potential theory which presented at a sufficiently modern level the "analytic" part of the theory relating to concrete kernels. This book remedied that deficiency in the literature. Although the book is directed to mathematicians who wish to be introduced to potencial theory for the first time, it will have interest also for specialists. It is important that knowledge of classical potential theory is not required for the reading of this book but the ideas and methods of modern function theory, functional analysis and general topology are necessary. The entire exposition is devoted to M. Riesz kernels and Green kernels. To justify such a selection of kernels the author says in the preface: "First, M. Riesz kernels include, as special (or limiting) cases, the classical Newtonian and logarithmic kernel. Second, changing the character of the singularity of a kernel leads to, form the point of view of analysis, very deep alterations of the theory: this is because the Laplace differential operator has to be replaced by a non-local integro-differential operator. With regard to Green kernels, they are essentially a model with which a potential theory for more general elliptic differential operators can be constructed."

The chapter headings are: Introduction, I. Potentials and their basic properties, II. Capacity and equilibrium measure, III. Sets of capacity zero. Sequences and bounds for potentials, IV. Balayage, Green functions, and the Dirichlet problem, V. Irregular points, IV. Generalisations (Pharagraps of this chapter: 1. Distributions with finite energy and their potentials, 2. Kernels of more general type, 3. Dirichlet spaces). Almost all references to the literature are at the end of the book under "Comments and Bibliographic References".

The book is well organized, the presentation of the material is concise but understandable, its format is nice.

I. Szalay (Szeged)

A. M. Olevskiĭ, Fourier Series with Respect to General Orthogonal Systems (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 86), VIII+136 pages, Springer Verlag, Berlin—Heidelberg—New York, 1975.

The last decade has been a period of intensive development in the theory of Fourier series. Advances have also been made in the theory of Fourier series with respect to general orthogonal systems. In particular, it was discovered that several results which had been seen to depend on the special peculiarities of the trigonometric system have in fact a considerably more general nature and are determined by such properties of ON systems as completeness or uniform boundedness.

The present book is primarily based on the investigations of the last fifteen years concerning Fourier series with respect to general ON systems. Results involving specific systems are examined only to the extent that they shed light on the problems of the general theory. The author does not touch at all upon the investigation of multiple Fourier series and spectral expansions, or upon multiplicative systems and other special classes of ON systems.

The fundamental results are given with proofs. However, the author has tried to avoid letting the technical details encumber the presentation. A number of the results of Chapters I and III were formerly only announced by the author and are now for the first time set forth in detail.

The main result presented in Chapter I is that there is no uniformly bounded ON system with respect to which every continuous function has an everywhere convergent Fourier series expansion. Thus the phenomenon of the local divergence of a Fourier series, discovered by du Bois—Reymond at the end of the last century, is connected not specifically with the trigonometric system, but has a general nature, it arises with any bounded ON system. A basic consequence is that no uniformly bounded ON system can form a basis in the space C of the continuous functions on a finite interval. Further, the uniform boundedness of an ON system determines a fixed order of growth of the Lebesgue functions $L_n(x)$. Namely, such systems always satisfy the relation $L_n(x) \neq o(\log n)$ on a set of positive measure. The proof of these results and others connected with them are based on a new method of estimating a lower bound, in the metric of the space L^1, for the partial sums of series of orthonormal functions.

Chapter II deals with condictions on the coefficients that ensure a.e. convergence. The general problem is to describe, for a given ON system φ, the class $\mathfrak{S}(\varphi)$ of sequences $\{c_n\}$ of coefficients for which the series $\Sigma c_n \varphi_n$ converges a.e.. Only in rare instances is there an effective solution to this problem; one of these exceptions is the Rademacher system $\mathfrak{r}$, for which $\mathfrak{S}(\mathfrak{r}) = l_2$. It is therefore of interest to study the intersection (or the union) of the classes $\mathfrak{S}(\varphi)$ for various sets of ON systems. Special attention has been given to the class $\mathfrak{S}_\Omega = \bigcap_{\varphi \in \Omega} \mathfrak{S}(\Omega)$, where Φ is the set of all ON systems. This chapter presents, among others, Tandori's results based upon the further development of Men'shov's method, Garsia's theorem on that the terms of any orthogonal series from L^2 can be arranged in such an order as to make the series converge a.e., etc.

In the last few years it was discovered that the Haar system $\{\chi_n\}$ plays a specific role in the class of all complete ON systems. A method is presented in Chapter III that permits in a number of cases a reduction of a problem for an arbitrary complete ON system to the same problem for $\{\chi_n\}$. Roughly speaking, if a Fourier series divergence phenomenon occurs with the Haar system, then such a phenomenon is unavoidable for every complete ON system. For example, Ulyanov and the author proved, independently of each other, that for any complete ON system there exists a function from L^2 whose Fourier series after appropriate rearrangement of its terms diverges a.e.; $\{\chi_n\}$ has the smallest possible Banach constant under fairly general conditions on Banach spaces B of functions; either $\{\chi_n\}$ forms an unconditional basis in B or there does not exist any unconditional basis at all in this space. In the rest of this chapter the behaviour of the Fourier coefficients of continuous functions is studied, the local Carleman singularity is extended for any complete ON system and a veriety of related results is proved.

Chapter IV is devoted to the a.e. and mean convergence of Fourier series with respect to general ON systems, but in contrast to Chapter II, properties here are first stipulated on the function, and not on the coefficients of the expansion. Considerable attention is paid to the peculiarities of Fourier series in the spaces L^p, $p<2$. In this case new phenomena arise, which do not in the case of L^2-series. For example, there exists an ON system, closed in C, whose Lebesgue functions are uniformly bounded, and nevertheless, the Fourier series of some $f \in \bigcap_{p<2} L^p$ diverges a.e.; Garsia's theorem does not extend to the spaces L^p, $p<2$; there exists a function in $\bigcap_{p<2} L^p$ whose Fourier series, with an appropriate ordering, can represent any measurable function, etc.

The book has been carefully and accurately written. The presentation is concise but always clear and well-readable. The main goal of the author is to survey the subject as it exists today, and it is perhaps not exaggerated to assert that this goal has been completely attained. It will certainly indicate the weak and strong spots in the edifice of the theory built so far, and thereby facilitate future research.

F. Móricz (Szeged)

Proceedings of the Symposium on Complex Analysis, Canterbury, 1973 (London Mathematical Society, Lecture Note Series 12), Edited by J. Clunie and W. K. Hayman, VII+180 pages, Cambridge University Press, 1974.

Part I, containing the contributions, mostly in short abstracts, of the participants, gives an interesting cross-section of some of the domains of present research in Complex Analysis. In the shorter Part II., W. K. Hayman gives a report on the progress on problems stated at a previous conference in the same area, at Imperial College, London, in 1964, and lists new problems that arose from the present symposium.

J. Terjéki (Szeged)

Sergiu Rudeanu, Boolean functions and equations, XIX+442 pages, Amsterdam—London—New York, North-Holland—American Elsevier Publ. Co., 1974.

Since the thirties the theory of Boolean algebras has developed in two very different lines: in a set-theoretical and in an algebraic direction. The set-theoretical approach, which goes back to Stone's representation theory, has become well-known and well-developed during the last decades and now there are three excellent monographs (those of Sikorski, Dwinger and Halmos) concerning this subject.

Curiously enough, the older algebraic approach, the theory of Boolean functions and of solutions of Boolean equations, which was intensively studied by Boole himself, Peirce, Poretski, Schröder, Löwenheim and other outstanding mathematicians of the last century, is much less familiar in the present mathematics. Its development has become scattered in the twentieth century, although it has been investigated by the same intensity as previously. This undesirable situation led to the rediscovery of a number of results and it has become indispensable for the further development to summarize and unify the theorems on this line into a "homogeneous" theory. This is done in the book of professor Rudeanu, one of the eminent specialists in this field.

The book consists of two parts.

Part I is doveted to the abstract theory of Boolean (systems of) equations and includes fundamental theorems concerning among others the solvability of Boolean equations, orthonormal solutions, symmetric equations, Boolean ring equations, Boolean transformations, parametric equations, syllogisms, Boolean arithmetic, Boolean geometry and Boolean calculus.

Part II, which is written in an informal style, deals with applications to switching theory.

Both parts are self-contained. For this reason the second part includes a brief summary of the first one. The book is completed by a bibliography consisting of more than 350 items.

This is a basic book for anybody who studies Boolean equations or would like to understand the mathematical foundations of switching theory.

András P. Huhn (Szeged)

Robert M. Switzer, Algebraic Topology — Homotopy and Homology (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band, 212) XII+526 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1975.

In the last few years several excellent textbooks have appeared on algebraic topology. Of these in the author's opinion the most notable is the *Algebraic topology* of E. H. Spanier. The present book offers more than any of these: it brings the reader to a point from which he can "begin research in certain areas of algebraic topology: stable homotopy theory, K-theory, cobordism theories". Of course it does not try to achieve the same very advanced level in all areas of algebraic topology, the choice is heavily influenced by the author's research interests. Although it goes considerably further than Spanier's book, there is a certain overlap between this and Spanier's book — especially in Chapters 0—6, 14 and 15 — thus Spanier's book is recommended by the author as a companion volume to his one.

The book under review has grown out from courses given by the author at the University of Manchester in 1967—1970, at Cornell University in 1970—1971 and at the Georg August University, Göttingen, in 1971—1972. It assumes the knowledge of the rudiments of algebraic topology, including singular homology, the fundamental group and covering spaces, and Chapter 12 also assumes some familiarity with differentiable manifolds.

The book is divided into 21 chapters. Chapters 0 and 1 contain some results from set-theoretic topology which are repeatedly used in the text and the basic definitions of category theory. Chapter

2 introduces the sets [X, Y] of homotopy classes of maps $f: X \to Y$ and studies the question of when is [X, Y] a group, when is a sequence $[X, W] \xleftarrow{f^*} [Y, W] \xleftarrow{g^*} [Z, W]$ exact, etc. Chapter 3 specializes to $X = S^n$, the n-sphere and considers $\pi_n(Y, y_0) = [S^n, s_0; Y, y_0]$, the nth homotopy group of $Y (n \geqq 1)$. The more elementary properties of these groups are proved in this chapter. In Chapter 4 the notions of fibration, weak fibration and fibre bundle are introduced and it is shown that every fibre bundle is a weak fibration. Examples of fibre bundles are given. The homotopy groups of S^n and $T^n = S^1 x \dots x S^1$ are computed using the observation that a covering $p: X' \to X$ is a fibre bundle with discrete fibre. Chapter 5 gives the notion and some straightforward properties of CW-complexes. Chapter 6 contains some finer homotopy results on CW-complexes: $\pi_n(X, x_0)$ depends only on the cells of dimension at most $n+1$; the suspension homomorphism $\Sigma: \pi_q(X, x_0) \to \pi_{q+1}(SX, *)$ $([f] \to [1 \wedge f])$ is an isomorphism for $q < 2n+1$ if X is an n-connected CW-complex, etc.. Chapter 7 turns from homotopy theory to homology and cohomology theories. It introduces the notion of generalized homology theory by means of the first six Eilenberg-Steenrod axioms and studies the direct consequences of these axioms. Chapter 8 shows how to construct homology and cohomology theories. In Chapter 9 it is shown that in Chapter 8 all possible cohomology theoreies on the category of CW-complexes have been constructed. In Chapters 10, 11 and 12 three important example are given: ordinary homology, K-theory and bordism. Chapter 13 is devoted to the study of product in homology and cohomology. The next chapter applies products to duality and orientability questions. In Chapter 15 comes the introduction of spectral sequences and the succeding chapter is concerned with characteristic classes. The headings of the last four chapters read as follows: Cohomology Operations and Homology Cooperations; The Steenrod Algebra and its Dual; the Adams Spectral Sequence and the *e*-Invariant; Calculation of the Cobordism Group.

The book's bibliography does not pretend to be comprehensive, since this is unnecessary because of the existence of Steenrod's compendium of all mathematical reviews related to topology. Instead, the bibliography has two goals: "(1) to suggest to the student where he might begin to pursue a given topic further and (2) to acknowledge the sources from which much of the material ... is drawn". A subject index is also given.

J. Szűcs (Szeged)

Robin J. Wilson, Introduction to graph theory, VIII+168 pages, Edinburgh, Oliver and Boyd, 1972.

A decade ago there existed 2—3 textbooks on graph theory; this number has grown very rapidly in the last years and books at different levels — introductory and advanced, general and special, "pure" and "applied" — have been written.

This book is intended to serve as "an inexpensive introductury text on the subject, suitable both for mathematicians taking courses in graph theory and also for non-specialists wishing to learn the subject as quickly as possible". Although several other books would meet this program, the present one is certainly one of the most successful attempts inasmuch its relative shortness is coupled with a fortunate selection of non-trivial concepts, results and applications. Wilson manages to avoid the dangers writing about such a broad and widely applicable subject. He not only defines the basic notions and illustrates them by well-chosen examples (this could have resulted in a book showing graph theory as a mere language without own mathematical contents) but also states those basic results which now should belong to the arsenal of anyone wishing to apply graph theory. On the other hand, he does not go into those proofs which are too long or too technical.

The book covers the following topics: Eulerian and Hamiltonian graphs (finite and infinite); trees and their enumeration; planarity and duality (stating Kuratowski's characterization but proving only Whitney's); coloring of graphs and chromatic polynomials; digraphs with applications to Markov chains; matchings (König-Hall Theorem); network flows and Menger's Theorem; matroids (describing the spectacular results on graphical representation of matroids). There are about 250 exercises, where several other related problems are touched.

Most of the material is presented in a neat, enjoyable way. The introductory chapters, with many well-chosen examples, are particularly well-written. I have found the chapter on Whitney-duality (§ 16) confusing; this notion is equivalent to abstract-duality of the preceding paragraph but this is explicitly mentioned in the proof of Theorem 16.c only. Instead of speaking about 3 kinds of duality, to state Whitney's definition as a characterization of the abstract-dual would have made this chapter much clearer.

Summarizing, this book is a well-written introductory text for those wishing to learn the basics of graph theory in such a way (the only reasonable way, in my opinion) that they also want to get a picture on how proofs, conjectures, notions, applications arise in this field.

László Lovász (Szeged)

T. Yoshizawa, Stability theory by Liapunov's second method (Publications of the Mathematical Society of Japan, Volume 9), VII+223 pages, The Mathematical Society of Japan, 1966.

T. Yoshizawa, Stability theory and the existence of periodic solutions and almost periodic solutions (Applied Mathematical Sciences, Volume 14), VII+233 pages, New York—Heidelberg—Berlin, Springer-Verlag, 1975.

The direct or second method of Liapunov became one of the most important tools of the qualitative theory of ordinary differential equations. It is known that several properties of the motions of conservative mechanical systems follow from the law of the conservation of energy This observations is generalized in Liapunov's second method: from the properties of a suitable vector-scalar function there follow certain properties of the motions of a general dynamical system without explicit knowledge of the motions. Many extensions, refinements, applications on stability theory, on the asymptotic behaviour and boundedness of solutions, etc., show that the theory built on this method has grown with a splendid speed. This method plays an important role also in the theory of control systems, dynamical systems, and functional-differential equations.

The author reached important results in the development of this method and also in its applications in various problems (e.g. in the characterization of the boundedness of various types by Liapunov functions; in the study of the stability and the boundedness of the solutions of functional-differential equations; in the discussion of the asymptotic behaviour and stability properties for periodic and almost periodic systems). Naturally, these topics of the stability theory play a central role in both monographs. But at the same time the first book and the second chapter of the second book give a good survey on the development and main results of Liapunov's second method.

The second monograph originates from a seminar on stability theory given by the author at the Mathematics Department of Michigan State University during the academic year 1972—73. As an introduction we get a guide for properties of almost periodic functions with parameters as well as for properties of asymptotically almost periodic functions. In the most interesting part of the book the converse theorem on integrally asymptotic stability and the relationship between total stability and other types of stability are treated including the newest results. Then the existence of a periodic solution in a periodic system is discussed in connection with the boundedness of solutions, and the existence of an almost periodic solution in an almost periodic system is considered in connection with some stability property of a bounded solution. Finally, sufficient conditions for the existence of a unique uniformly asymptotically stable periodic (almost periodic) solution of a periodic (almost periodic) system are proved by the aid of Liapunov functions.

The books are very useful and important for everybody interested in the qualitative theory of ordinary differential equations, especially in the applications of Liapunov's second method.

Chapter headings *(first book)*: Preliminaries; Liapunov stability and boundedness of solutions; Extensions of stability theory; Extreme stability and stability of sets; Converse theorems on stability and boundedness; Perturbed systems; Existence theorems for periodic solutions and almost periodic solutions; Functional-differential equations; *(second book)*: Preliminaries; Stability and boundedness; Existence theorems for periodic solutions and almost periodic solutions.

L. Hatvani—L. Pintér (Szeged)

Helmut H. Schaefer, Banach Lattices and Positive Operators (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 215), XI+367 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1974.

The end of the author's earlier book "Topological Vector Spaces" contains a brief summary of results in topological vector lattices. Since the first edition of that book, the theory of Banach lattices developed into an independent theory, though it has interesting applications to functional analysis, and its results can be applied even in the theory of topological vector spaces. This situation is well reflected in the present book.

The first chapter deals with positive complex matrices to supply a unified discussion of the most important operator theoretic properties of positive matrices and to serve as a motivation for the study of positive operators on Banach lattices. The basic properties of Banach lattices are treated in Chapter II; some special Banach lattices are also considered. The purpose of the next chapter is to look for a representation theory of Banach lattices, similar to that of commutative Banach algebras. A beautiful result is a generalization of the Halmos-von Neumann theorem on ergodic dynamical systems. Each section of the book treats very interesting questions, as tensor products of Banach lattices, lattices of operators between Banach lattices, Hilbert lattices, the peripheral spectrum of positive operators, to mention only some of them.

The style of the author is the same as in his earlier book; the presentation of the material is concise but easily readable, the notations are suggestive and exact. We can surely assert that this book is unique nowadays.

T. Matolcsi (Budapest)

Roger Temam, Analyse numérique (Collection SUP "Le Mathématicien", 3), 119 pages, Paris, Presses Universitaires de France, 1970.

La résolution approchée des équations fonctionnelles constitue une partie importante de l'analyse numérique. Ce livre sert d'une bonne introduction à la théorie des résolutions approchées et il donne aussi des exemples pratiques. La première partie traîte de quelques aspects de l'approximation de la solution d'une équation de type elliptique, par exemple le théorème de Lax-Milgram et la méthode de Galerkin, le problème de l'approximation par des éléments non appartenant à l'espace considéré, et l'estimation de l'erreur. Les résultats sont appliqués aux problèmes de Dirichlet et de Neumann.

Le livre est bien construit et on peut le lire aisément. La lecture ne suppose qu'une certaine connaissance des espaces hilbertiens, éléments de la théorie de la mesure et de la théorie des distributions.

T. Matolcsi (Budapest)

Wolfgang Walter, Einführung in die Theorie der Distributionen, VIII+211 Seiten, Mannheim—Wien—Zürich, Bibliographisches Institut—Wissenschaftsverlag, 1974.

Die Theorie der Distributionen ist von grosser Wichtigkeit sowie in der reinen als auch in der angewandten Mathematik. Heutzutage gibt es viele guten Bücher, die eine Einführung in diese Theorie bieten. Man kann auch diese Buch zu ihnen zählen. Wenn mann ein solches Buch in der Hand hält, sucht man, was es von den anderen unterscheidet. Dieses Buch kann leicht und mit Interesse gelesen werden und sein letzter Paragraph hat eine kurze Einführung in die Theorie der Sobolev-Räume zum Inhalt. Leider werden die konventionellen Bezeichnungen, die nicht konsequent und manchmal auch irreführend sind, gebraucht, obwohl man schon ein ausgezeichnetes Bezeichungssystem für Distributionen vorhanden hat.

T. Matolcsi (Budapest)

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